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Department of Mathematical Economics, Econometrics,
Financial and Insurance Mathematics

Master thesis

SYSTEM DYNAMIC MODELING OF KNOWLEDGE ACCUMULATION PROCESSES

Done by: student of the MTE-61m
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Introduction

We will discuss different perspectives on how the allocation of resources for knowledge production is determined and what types of knowledge are available.

Variable "A" is obtained as knowledge in R&D. However, education can be in many forms. Knowledge may be very applied or very abstract and it is important to think about these types of knowledge. At the highest level, these are the most widely used basic scientific discoveries, such as the Pythagorean theorem. On the other hand, there are some products that know how to operate a lawn mower on a cold morning. Between them there are many ideas - from the design of a transistor or the invention of a rotating table to the kitchen of a fast food restaurant to a recipe for soft drinks.

A lot of these different types of knowledge have impact on economic growth. Suppose that for 100 years the basic scientific progress has stopped, or the inventions that are now applied to many goods, or the new design of something that is now applied. All this could affect growth in the economy, and would most likely have a negative impact, in any case, all these changes would lead to a decrease in growth.

Obviously, the determinants of the accumulation of different types of knowledge will be different. For example, the strengths of basic math are different from those that outperform catering restaurants. Accordingly, we cannot expect that there is only one theory of knowledge growth. However, we can find the main factors that affect the accumulation of knowledge. But all different types of knowledge have a common necessity - it is inanimate. This means that absolutely everyone is free to use one or another theorem in mathematics, or any recipe for cooking. The opposite is any private economic goods, which in turn are competitive: for example, only one person can use clothes at a time.

From this common feature of different types of knowledge, it follows that competitive market forces cannot completely control the distribution and production of knowledge. The marginal cost of such a function as providing the subject of knowledge to another user will be zero. As a result, in a competitive market, the cost of renting knowledge is also zero. Therefore, there is no motive to create knowledge for the purpose of private economic gain. From this we get two options: first, knowledge is bought for exceeding marginal costs, and second, development has no motivation of market forces.

However, there is an exception. If it is possible to prevent someone from using the product, then such a product is exceptional. For example, clothing: the owner of the item of clothing prevents others from using it.

Such uniqueness can be among the knowledge. It depends on the nature of knowledge, as well as on such economic institutions that regulate all property rights. Patent laws will be a good example, they give inventors the right to use their own discoveries and devices. It is these laws that allow the owners of inventions to prevent other people from using these inventions. On the other hand, copyright laws do not fully protect textbook owners from plagiarism in the organization of the textbook. Copying the entire textbook is prohibited, but the law may not prohibit other authors from revising or improving the organization of the textbook.

Exclusivity is mainly influenced by the type of knowledge and not the legal system. For example, the recipe for famous drinks such as "*Fanta*" is complex enough not to use a patent or copyright for it. However, the technology of recording on a video camera is simple, and the authors of, for example, television shows can not prevent others from recording these television shows and the "*knowledge*" contained therein.

Thus, the degree of exclusivity affects how the distribution and development of knowledge differs from perfect competition. If some type of knowledge is not exceptional at all, the authors of this knowledge will not receive private property in their development. In contrast, when the type of knowledge is exceptional, you can license the right to use that knowledge and get positive feedback, for example from research.

Chapter 1

Framework and Assumptions

Here we present and analyze a model in which there is an endogenous distribution of factors of economic production between the accumulation of knowledge and other activities. This will happen simultaneously with Solow's model in which the economy is treated. Let's discuss the dynamics of the economy and the determinants of long-term growth. Consider the assumptions of how the population is implemented for such a purpose as adding knowledge.

1.1 Overview

To begin with, we will present a separate sector of the economy, in which new models of knowledge accumulation are created. To do this, we simulate the distribution of resources between two sectors: the first is the sector in which products are usually produced, and the second is the new R&D sector, as well as the method of generating new ideas through R&D.

In our modeling we will consider the production of new technologies from a mechanical point of view. To do this, we describe the following function of production: capital, accumulation of labor and technology will be combined deterministically to form technological improvements. To model technological progress, we will add a little extra information, because we are interested in growth directly over long periods.

At any time, we can add a shift parameter to the production function and thus investigate the impact of changes in this parameter if we are interested in assessing the impact of modifications on the achievement of R&D.

Let's make some more relief. One of them is the production of goods and R&D are the generalized production functions of Cobb-Douglas. The sum of the indicators is not necessarily limited to one, but it is still a power function. And the second simplification is that, as in the Solow's model, the share of retained output and labor fraction is exogenous and constant in the resource and development sector. These simplifications have not changed the main consequences of the model.

1.2 Model Specification

If we reduce the research and development model, as well as the growth models created by Romer (1990), Aghion and Howitt (1992) and Grossman and Helpman (1991a), we get this model. The basis of this model are the following main variables: capital (K), labor (L), technology (A) and production (Y). It is continuously configured. There are two sectors: the first is the one in which products are produced (production of goods and services), and the second sector is responsible for creating new technologies (R&D). The production of goods uses the following fraction: $1-a_L$, and the R&D industry uses the labor fraction. Everything except the share of a_K capital is used for the production of new goods, and the share of a_K capital is used for resources. Using certain knowledge or concept in one area allows you to use it in another area. It follows that these two sectors have direct access to the spectrum of knowledge A. Let the variables a_L and a_K be exogenous and constant.

Equation 1.1 calculates the amount of outputs for time t,

$$Y(t) = [(1-a_K)K(t)]^{\alpha}[A(t)(1-a_L)L(t)]^{1-\alpha}, \quad 0 < \alpha < 1, \tag{1.1}$$

Where in the production sector: α represents the elasticity of capital, and $1-\alpha$ - the elasticity of the efficient labor force. Three times the input triples the amount that can be obtained, because function 1.1 is a Cobb-Douglas production function with a constant return to scale. The development of new technologies depends on the amount of capital and labor involved in the development and research sector.

Given the hypothesis of generalized production of Cobb - Douglas, we make the following:

$$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta}, \quad B > 0, \quad \beta \geqslant 0, \quad \gamma \geqslant 0, \tag{1.2}$$

here, θ is the elasticity of knowledge, γ is the elasticity of labor, β is the elasticity of capital in the development sector. And parameter B is the offset, which means that the growth of technology depends on previous existing resources (this is the same as using all the parameters in equation 1.2 at time $t-1$). Let the knowledge function not be considered a constant return to scale. If the inputs are doubled, these new inputs can do exactly what the old ones did, doubling the number obtained, which is a standard argument for constant returns. However, if we are talking about the production of knowledge, then the exact repetition of the current input will make the same set of discoveries twice. The influence on the achievement of R&D due to the available stock of knowledge is reflected by the parameter θ.This effect is performed in two directions: first, previous discoveries can give tools and ideas, then $\theta > 0$; and the second is that in the beginning you can create the simplest discoveries. In the second situation, it is difficult to create new discoveries when the stock of knowledge is greater than zero, and therefore $\theta < 0$. It follows that in equation 1.2 θ can have any value. Then the new capital depends on the volume of production, capital stock, savings rates and depreciation rates. For convenience, the depreciation rate = 0, and the savings rate is exogenous and constant. We obtain a description of new knowledge in equation 1.3.

$$\dot{K}(t) = sY(t). \tag{1.3}$$

Simplifying, suppose that the population accumulates at a constant rate of growth. It can't be negative, therefore

$$\dot{L}(t) = nL(t), \quad n \geqslant 0. \tag{1.4}$$

Initial values of L, A and K are greater than zero. We show in the following diagram the relationship between all variables.

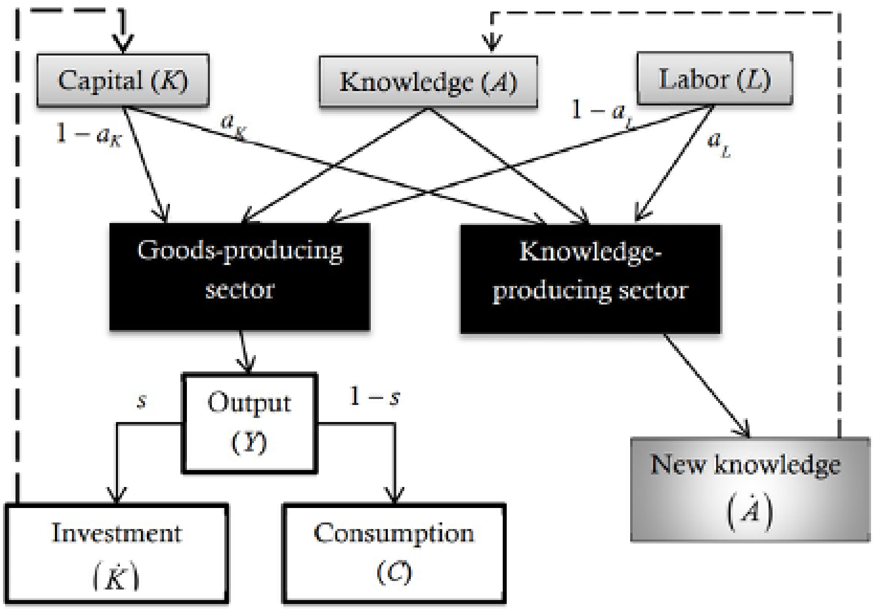

Figure 1.1: Diagram of goods and knowledge production sectors

Chapter 2

Model without Capital

In this chapter we will discuss the model without capital. We will also investigate the influence of knowledge elasticity (θ on growth rates. Let us investigate the following cases, which may be the value of θ: when $\theta < 1$, $\theta > 1$ and $\theta = 1$. And decide in which of these cases long-term growth is possible.

2.1 Dynamics of Knowledge Accumulation

The production function of the model (equation 1.1) in which there is no capital will look like this

$$Y(t) = A(t)(1 - a_L)L(t) \tag{2.1}$$

Similarly, the function of producing new knowledge (equation 1.2) is simplified to

$$\dot{A}(t) = B[a_L L(t)]^{\gamma} A(t)^{\theta}. \tag{2.2}$$

Equation 2.1 shows that the growth rate per worker is equal to A and the yield per worker is also equal to A. Let's focus on the dynamics of A in equation 2.2. The growth rate of A (g_A) is determined by this equation

$$g_A(t) = \frac{\dot{A}(t)}{A(t)} = B a_L^{\gamma} L(t)^{\gamma} A(t)^{\theta - 1}. \tag{2.3}$$

Let us simplify this by taking on both sides, and express it in relation to time. We obtain *growth rate of the* g_A:

$$\frac{\dot{g}_A(t)}{g_A(t)} = \gamma n + (\theta - 1) g_A(t). \tag{2.4}$$

After multiplying what turned out by 2.4, we obtain the following equation of the derivative of the growth rate

$$\dot{g}_A(t) = \gamma n g_A(t) + (\theta - 1) g_A^2(t). \tag{2.5}$$

Initial values of knowledge accumulation (A) and labor (L), exogenous parameters γ, θ, B of the model determine the initial value of $g_A(t)$.

Using equations 2.1, 2.2, 2.3 and 2.5, we will build a dynamic model without capital in the software program Stella Architect. We will investigate the growth rate of output using this model.

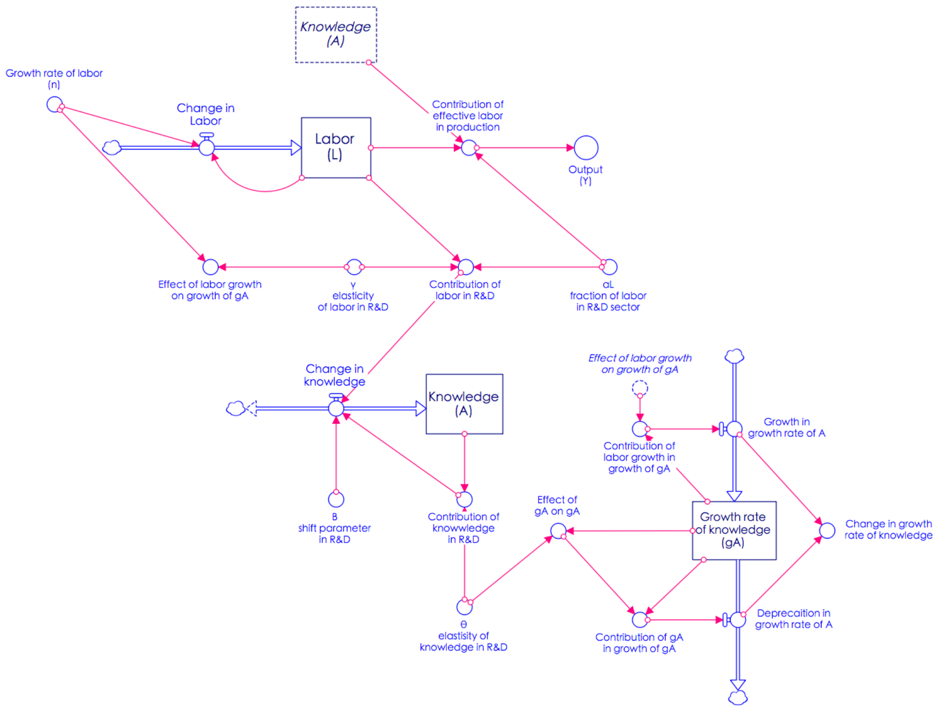

Figure 2.1: Stock and flow diagram of the model without capital

Let us investigate the growth of the model for different values of the elasticity of knowledge θ ($\theta < 1$, $\theta > 1$, and $\theta = 1$).

We must investigate the growth of the model for different values of elasticity of knowledge θ ($\theta < 1,\ \theta > 1$, and $\theta = 1$).

Case 1: $\theta < 1$

The phase diagram for g_A is shown in Figure 2.2, which indicates that $\theta < 1$.

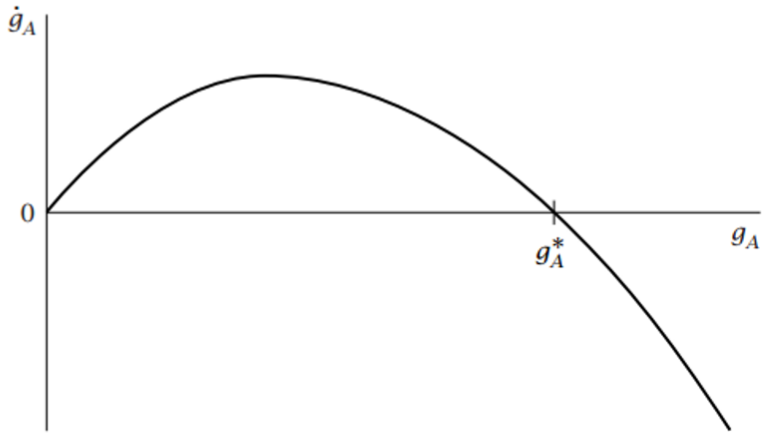

Figure 2.2: Phase diagram for growth of knowledge $\theta < 1$

In this case, $\dot{g}_A$ is displayed as a function A. The production function for knowledge takes into account only the positive values of g_A because g_A is grater then zero, as shown in equation 2.2. Equation 2.5 illustrates that for $\theta < 1 \rightarrow \dot{g}_A > 0$ for small positive values of g_A and $\dot{g}_A < 0$ for large values, as shown in Figure 2.2. Denote the value at the extreme point g_A as g_A^*, we obtain that $\dot{g}_A = 0$. We derive yields $\gamma n g_A^* + (\theta - 1) g_A^{*2} = 0$ from equation 2.5 and $\dot{g}_A = 0$, divide it by and g_A^* and the solution of this equation with respect to g_A^* gives us this

$$g_A^* = \frac{\gamma}{1-\theta} n \tag{2.6}$$

We can say that g_A converges to g_A^*, and this does not depend on the initial conditions. When $g_A(0) < g_A^*$, then $\dot{g_A} > 0$, and g_A increases until it reaches a stable point g_A^*. When $g_A(0) > g_A^*$, then g_A decreases to reach g_A^*. A and Y/L increase steadily with the rate g_A^* when g_A reaches g_A^*.

Looking at the model, we can conclude that the growth rate g_A^* is an increasing function of population growth rate n. Favorable population growth has an important impact on the sustainable growth of production per employee. For example, in countries with faster population growth, the increase in production per worker is not higher on average.

It follows from equation 2.6 that long-term growth is not affected by the labor force in the resources and development sector. This may seem wrong, because development depends on technological progress, and technological progress in turn is endogenous. It is expected that the growth of the workforce dedicated to technological progress will lead to long-term development. To understand this, consider a situation where a_L increases from the point where A is rising at rate g_A^*.

a_L is not in equation 2.3. Therefore, the increase in a_L does not affect the behavior in the graph of $\dot{g_A}$ as a function of g_A. Despite this, a_L is included in the expression 2.3, $g_A : g_A(t) = \frac{\dot{A}(t)}{A(t)} = Ba_L^{\gamma}L(t)^{\gamma}A(t)^{\theta-1}$. In Figure 2.3 we can investigate that when a_L increases or decreases, then it does not affect the stable point of growth rate.

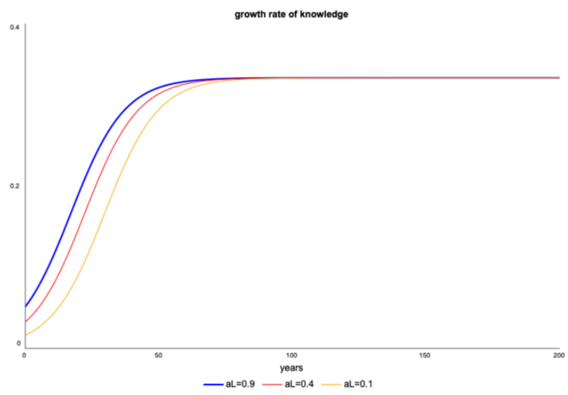

Figure 2.3: The effects of an increase in a_L on g_A when $\theta < 1$

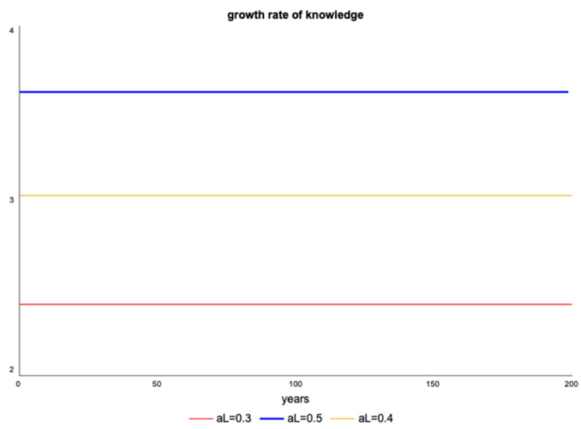

Figure 2.4: The effects of an increase in a_L on g_A when $\theta = 1$

We get g_A instantly increases when a_L increases but it does not affect $\dot{g}_A$. Note that the growth rate of knowledge is not supported. When $g_A > g_A^*$, then $\dot{g_A}$ is less than zero and g_A goes to the equilibrium g_A^*. The situation when $\theta < 1$, means that the available stock of knowledge does not contribute to full support for the development of new knowledge.

Combining the effect of increasing the coefficient of savings on the production route in the Solow model, means that the increase in a_L leads to an increase in g_A with a subsequent gradual return to the initial level.

Case 2: $\theta = 1$

The case when $\theta = 1$, the current information is quite productive in creating new knowledge. Namely, new knowledge is proportional to the stock.

g_A increases with time when there is a positive population growth, it follows that the dynamics of the model is similar to the case $\theta > 1$ (this case will be discussed below). From another point of view, when population growth = 0, g_A is constant regardless of the initial values. There is no adaptation to a balanced development path. The economy will continue to expand no matter where it starts.

It is normal to assume that products are consumed completely despite the fact that they are produced in this economy only for consumption. Based on the fact that the goods produced in this economy have only one use - consumption, it is logical to think of it as full consumption. Expression $1 - a_L$ is part of society's resources allocated to the production of goods for current consumption, while a_L is the part stored for the production of goods that will be useful for the production of products in the future. In summary, we can say that a_L acts as a measure of the level of savings in this economy.

$$g_A(t) = Ba_L^{\gamma}L(t)^{\gamma}, \tag{2.7}$$

$$\dot{g}_A(t) = \gamma n g_A(t). \tag{2.8}$$

Analyzing the case $\theta = 1$ and $n = 0$, we can say that it offers a straightforward model of the model, when long-term growth affects the rate of savings. Models of this type are defined as linear growth models that have earned a lot of advertising in working on endogenous growth due to their simplicity.

Case 3: $\theta > 1$

Consider the third case - when $\theta > 1$. This is similar to a situation where new knowledge is always more than the current stock.

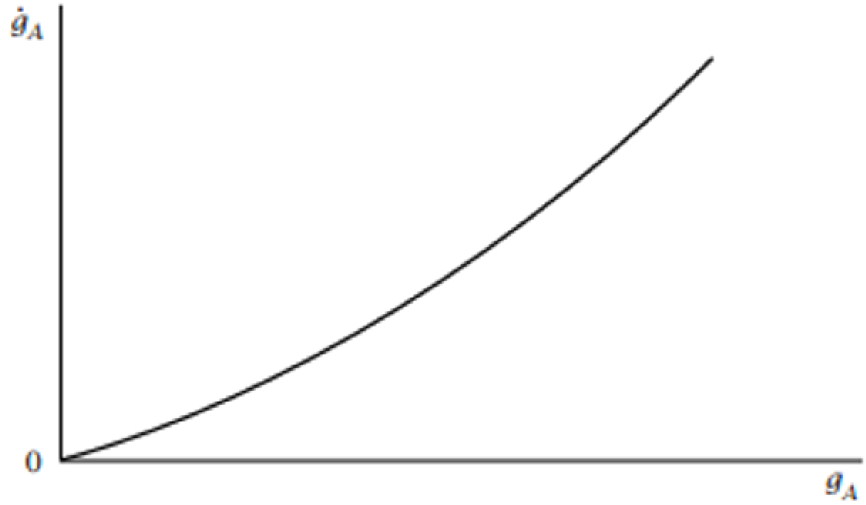

Figure 2.5: The dynamics of the growth rate of knowledge when $\theta > 1$

From formula 2.5 it follows that when θ becomes greater than 1, $\dot{g_A}$ is positive for all g_A. $\dot{g_A}$ increases in g_A - this can be seen in Figure 2.5.

At this stage, the chart shows that instead of moving towards a balanced path of growth, economic growth is growing. Knowledge in the production of new knowledge is so useful that every small rise in its stage leads to much more new knowledge. So, when the accumulation of knowledge begins, just by model, then economic growth begins.

There is a significant effect of increasing the share of the labor force involved in research and development. As before, due to the increase in equation 2.3 leads to instantaneous increments of g_A. The value of g_A increases as the role of g_A increases. And the faster g_A grows, the faster its growth rate increases. As a_L increases, the growth rate$A$ exceeds the otherwise increasing level.

2.2 Effect of Returns to the Scale of Produced Factors

The explanation for the fact that all three different cases have such a different effect is that if θ is lower or higher than or equal to 1, it determines whether the scale of production factors decrease, increase or consistency.

Employment growth is exogenous, and the model is without capital, so knowledge is the only factor influencing growth. Knowledge returns constantly in the production of goods. It follows that the return to knowledge in the field of knowledge production will be defined as the return on the scale of knowledge. To understand why the return on production is critical to economic behavior, we make two assumptions: the first that the industry is on a certain path, and the second that the exogenous rise of 5 percent increases by A.

Case 1: $\theta > 1$, $\dot{A}$ will be grow by more than 5 percent. The growth rate A increases for this case. Case 2: $\theta = 1$, $\dot{A}$ increases by 5 percent. Knowledge is productive enough to gain new knowledge that growth A is independent. But an increase in A does not affect its growth rate. Case 3: $\theta < 1$, $\dot{A}$ increases less than 5 percent, and the growth rate A decreases.

2.3 Effect of Population Growth

The model without capital in the case when θ is less than 1 is implicit in that positive population growth is necessary to achieve long-term growth in per capital income. The case θ is equal to 1 and when $n = 0$, the growing function of the population is long-term growth. When θ is greater than 1, then an increase in population growth leads to an increase in income per capital. To understand these results, consider the accumulation of knowledge in equation 2.3. The equation $g_A(t) = Ba_L^{\gamma} L(t)^{\gamma} A(t)^{\theta-1}$. The perfectly natural idea built into this equation is that when more people make discoveries, more discoveries will be made. And when more discoveries are made, then the stock of knowledge grows faster, and therefore the output per person also increases faster. In the case θ is equal to 1 and $n = 0$ long-term growth increases at the population level. When θ more than 1 increase in population (or its growth rate) leads to growth.

When θ is less than 1, the return to scale decreases. In this situation, the production of new knowledge increases less than the current stock, despite the fact that knowledge can be valuable in creating new knowledge. Therefore, without other assistance in the creation of new knowledge, progress will slow down. Because people contribute to knowledge production, population growth provides something else: population growth is important for long-term growth, and population growth is increasing.

Chapter 3

Endogenous Economic Growth Model

In this chapter, we will look at the model in which we will introduce capital and find out how this affects the preliminary analysis. Also we will consider a different combination of the elasticity of knowledge and capital and explore in which cases a long-term nature is possible.

3.1 Dynamics of Knowledge and Capital

Let us focus on the dynamics of growth rates A and K in combination with the analysis of a simple model. Replacing the production function 1.1 with the function of capital accumulation 1.3 we obtain the equation for changing knowledge.

$$\dot{K}(t) = s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}K(t)^{\alpha}A(t)^{1-\alpha}L(t)^{1-\alpha}. \tag{3.1}$$

Defining $g_K(t) = \frac{\dot{K}(t)}{K(t)}$, $c_K = s(1-a_K)^{\alpha}(1-a_L)^{1-\alpha}$ and using equation 3.1 give us growth rate of capital:

$$g_K(t) = c_K \left[\frac{A(t)L(t)}{K(t)}\right]^{1-\alpha}. \tag{3.2}$$

Taking logs from both sides from equation 3.2 and differentiating with respect to time returns growth rate of $g_K(t)$:

$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t)+n-g_K(t)]. \tag{3.3}$$

When $g_A(t)+n-g_K(t)$ is positive g_K increases, negative g_K decreases, zero g_K constant. This information is illustrated in the figure below (Figure 4.18).

The division of the equation 1.2 by A on both sides reflects the growth rate of A.

$$\dot{A}(t) = B[a_K K(t)]^{\beta}[a_L L(t)]^{\gamma} A(t)^{\theta} \quad |/A$$

$$g_A(t) = c_A K(t)^{\beta} L(t)^{\gamma} A(t)^{\theta-1}, \tag{3.4}$$

where $c_A \equiv B a_K^{\beta} a_L^{\gamma}$.Equation is identical to equation 2.3 in the simple version of the model except for the expression $K^{\beta}(t)$. Taking logs on both sides and differentiating with respect to time returns the equation of growth rate $g_A(t)$:

$$\frac{\dot{g}_A(t)}{g_A(t)} = \beta g_K(t) + \gamma n + (\theta-1) g_A(t). \tag{3.5}$$

The situation when $\beta g_K(t) + \gamma n + (\theta-1) g_A$ is positive g_A and A rise, negative g_A and A fall, zero g_K constant and A rises constantly. We can see it in Figure 4.19. Looking at the production function for derivation in equation 1.1 we can see a constant return to the scale of two factors, capital and knowledge. And equation 1.3 illustrates that $\beta + \theta$ is the force of return to scale in knowledge and capital production, K and A increase by a factor of X, $\dot{A}$ by a factor $X^{\beta+\theta}$.

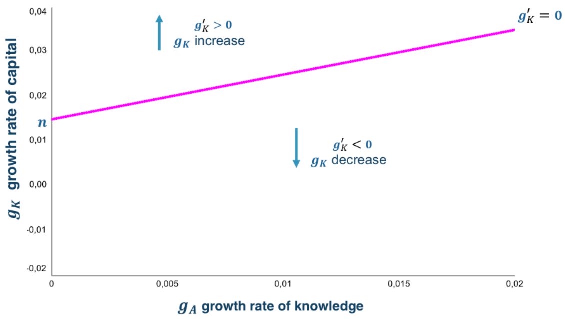

Figure 3.1: The dynamics of the growth rate of capital in the general version of the model

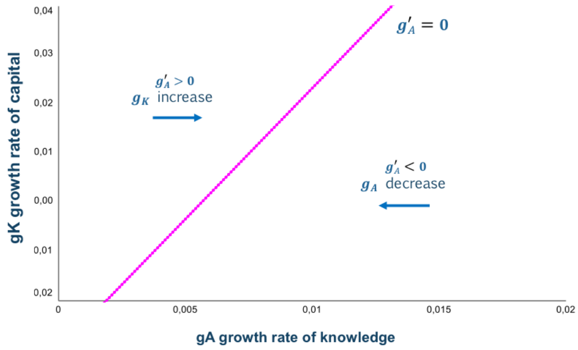

Figure 3.2: The dynamics of the growth rate of knowledge in the general version of the model

Case 1: $\beta + \theta < 1$

If $\beta + \theta < 1$ then $(1-\theta)/\beta > 1$. Thus the curve of $\dot{g}_A = 0$ is higher than the curve $\dot{g}_K = 0$. We can see this on the Figure 3.3. The graph shows that we have equilibrium point.

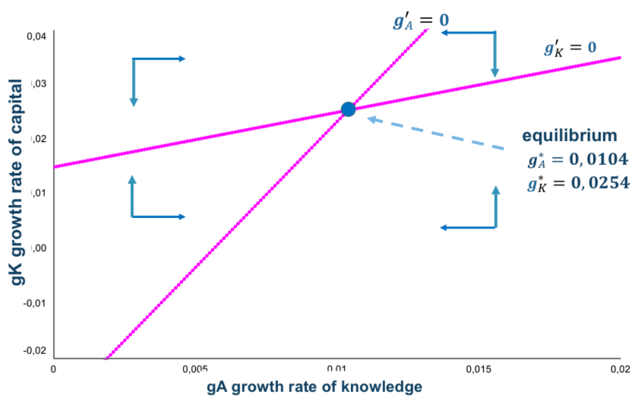

Figure 3.3: The dynamics of the growth rates of capital and knowledge when $\beta + \theta < 1$

The initial values of g_{A0} and g_{K0} are determined by the initial values of A_0, K_0 and L_0, as well as other parameters of the model. You can see in Figure 3.3 that where g_A and g_K begin, in the diagram they converge to B. At point B $\dot{g}_A = 0$ and $\dot{g}_K = 0$. The values of g_A and g_K at point B, denoted as g^*_A and g^*_K, therefore it is necessary to satisfy the following equations:

$$g_A^* + n - g_K^* = 0 \tag{3.6}$$

$$\beta g_K^* + \gamma n + (\theta - 1) g_A^* = 0 \tag{3.7}$$

Combining these equations we obtain

$$\beta g_A^* + (\beta + \gamma) n + (\theta - 1) g_A^* = 0 \tag{3.8}$$

which give us an equilibrium point for the growth rate of knowledge:

$$g_A^* = \frac{\beta + \gamma}{1 - (\theta + \beta)} n. \tag{3.9}$$

From the fact that $g_K^* = g_A^* + n$, in equation 1.1, when A and K increase at velocities g_A^* and g_K^*, respectively. Then the output increases at a rate equal to the rate of capital growth g_K^* and production per worker increases at a rate of g_A^*.

This situation is similar to the case when θ is less than 1, so long-term growth is a growing function of population growth and is zero , when population growth is zero and long-term economic growth is endogenous.

This model and a simplified version of this model, when $\theta < 1$, is called *semi-endogenous growth models*. Long-term growth occurs endogenously in the model, and also depends only on population growth and the parameters of the knowledge production function.

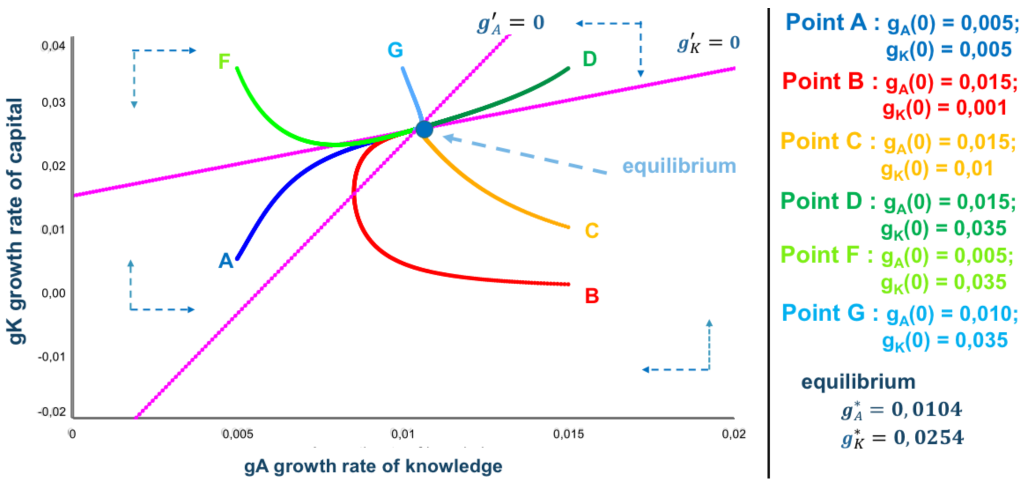

Figure 3.4: The dynamics of the growth rates of capital and knowledge when $\beta + \theta < 1$)

Let the capital elasticity in the research and development sector $\beta = 0.15$. And the elasticity of capital in the commodity production sector $\alpha = 0.3$. Knowledge $\theta = 0.2$. Labor elasticity in the research field $\gamma = 0.2$ and labor growth rate $n = 0.015$. Figure 3.4 presents the dynamics of capital and knowledge growth at $\beta + \theta < 1$ for different values of g_{A0} and g_{K0}. It is easy to see on the graph that it does not matter what the initial values are for g_{A0} and g_{K0}, because over time the growth rate will reach equilibrium points. For this case, there are equilibrium points

$$g_A^* = \frac{\beta + \gamma}{1 - (\theta + \beta)} n = \frac{0.15 + 0.25}{1 - (0.25 + 0.15)} 0.015 = 0.012$$
$$g_K^* = g_A^* + n = 0.012 + 0.015 = 0.027$$

Now consider in more detail the growth rate of knowledge with the above values for exogenous parameters of the model. Figure 3.5 shows that if $g_A < g_A^*$ or $g_A > g_A^*$ in the long run the growth rate of knowledge reaches equilibrium $g_A \to g_A^*$, where $g_A^* = 0.012$. Figure 3.6 shows the same behavior at the rate of capital growth. The rate of capital growth reaches $g_K^* = 0.027$. over time

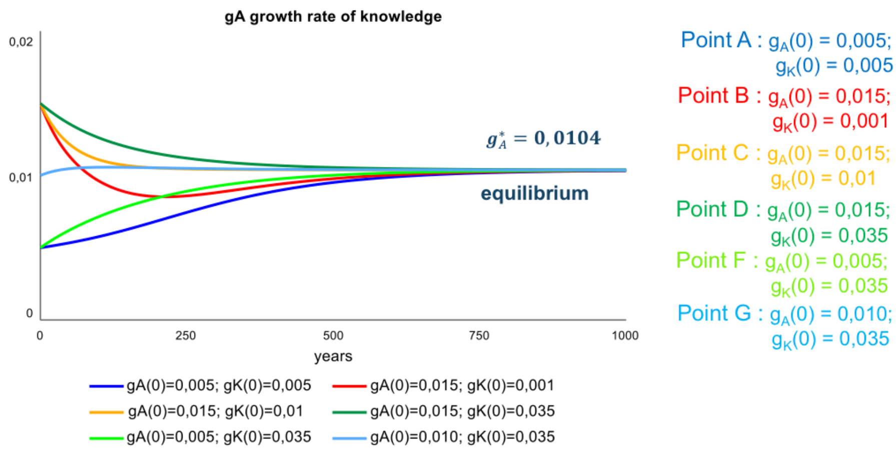

Figure 3.5: The dynamics of the growth rates of knowledge when $\beta + \theta < 1$

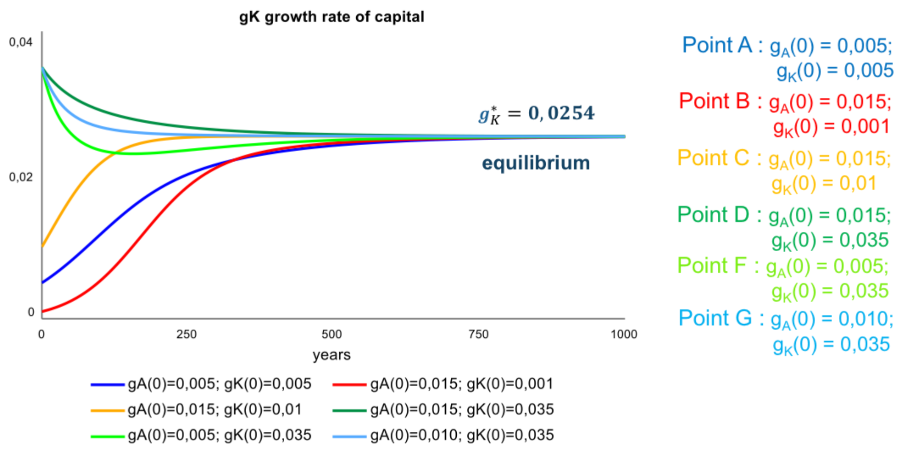

Figure 3.6: The dynamics of the growth rates of capital when $\beta+\theta<1$

Case 2: $\beta+\theta=1$ and $n=0$

Therefore, $\dot{g}_K=0$, when $g_K=g_A+n$ and $\dot{g}_A=0$, when $g_A=-(\gamma n/\beta)+[(1-\theta)/\beta]g_A$. The dynamics of g_A and g_K will be on a line with a slope of 45 degrees (this is shown in Figure 3.7), regardless of where the economy begins. Summarize, when $\beta+\theta=1$ and $n=0$, the expressions we gave earlier can be simplified to $g_K=g_A$.

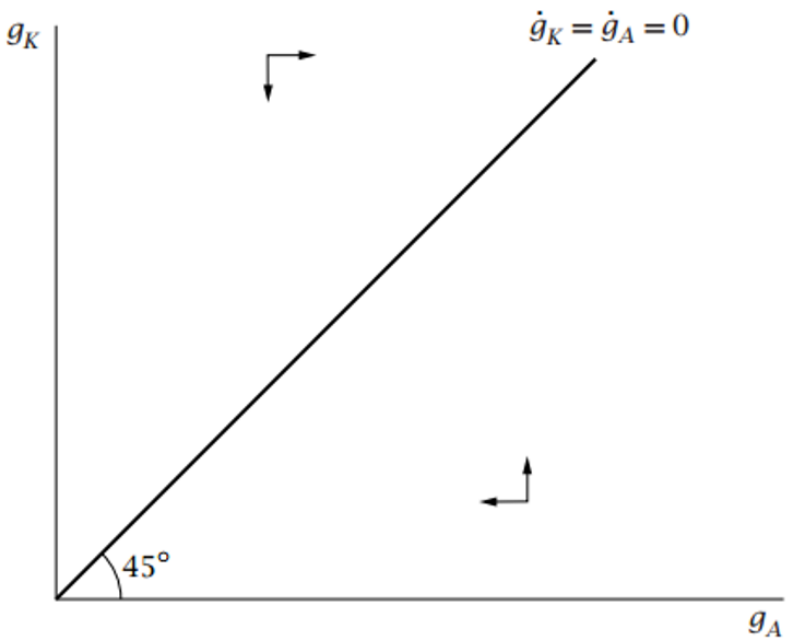

Figure 3.7: The dynamics of the growth rates of capital and knowledge when $\beta+\theta=1, n=0$

We can conclude that long-term growth depends on many parameters of the model. The models we considered in the previous section (when $\theta \geqslant 1$) and the model in this section, when $\beta + \theta > 1$ or $\beta + \theta = 1$ and $n > 0$ are endogenous growth models.

Chapter 4

Learning-by-Doing

The last determinant of knowledge accumulation differs in nature. The essence of this idea is that when people produce goods, then they are sure to think of ways to improve the production process. Consider the situation of the American economist Arrow (1962) who gives an empirical pattern after the introduction of a new aircraft design: the time (required to build the frame of the ultimate aircraft) is inversely proportional to the cubic root of the total number of aircraft already produced. Such an increase in labor productivity does not require explicit innovations in the production process. Thus, the accumulation of knowledge occurs in part as a side effect of normal economic activity. And not as a result of purposeful efforts. This type of accumulation of knowledge is called learning by doing.

The speed of knowledge accumulation depends on how much new knowledge is formed through ordinary economic activity (rather than the share of resources of the economy engaged in R&D) when learning in practice is a source of technological progress.

Therefore, the analysis of educational activities needs to be changed in our model. Since all input resources are engaged in the production of goods, the production function will have the form

$$Y(t) = K(t)^{\alpha}[A(t)L(t)]^{1-\alpha} \tag{4.1}$$

The situation when learning occurs as a side effect of the production of new capital is the simplest case of learning experience. Thus, the stock of knowledge is a function of the stock of capital because the increase of knowledge is a function of the increase of capital. Therefore, there is a single state variable. The power function will be

$$A(t) = BK(t)^{\phi}, \quad B > 0, \quad \phi > 0. \tag{4.2}$$

Equations describing the accumulation of capital and labor 1.3 - 1.4 together with 4.1 - 4.2 characterize the economy.

Substitute 4.2 into 4.1 in order to analyze this economy. We will receive

$$Y(t) = K(t)^{\alpha}B^{1-\alpha}K(t)^{\phi(1-\alpha)}L(t)^{1-\alpha} \tag{4.3}$$

From the fact that $\dot{K}(t) = sY(t)$ it follows that the dynamics of K is given by

$$\dot{K}(t) = sB^{1-\alpha}K(t)^{\alpha}K(t)^{\phi(1-\alpha)}L(t)^{1-\alpha} \tag{4.4}$$

In the model of knowledge accumulation without capital, the dynamics of A is given by $\dot{A}(t) = B[a_L L(t)]^{\gamma}A(t)^{\theta}$ in equation 2.2. The structures of the two models are the same compared to equation 4.4. In the second section, the model includes only a single productive contribution, knowledge. However, now we can think that there is capital - only one productive contribution. Equations 2.2 and 4.4 show that the dynamics of the two models are essentially the same. This means that we can use the results of our analysis of the previous model to analyze this one. Now the main factor in the dynamics of the economy is the comparison of θ with 1. Just as $\alpha + \phi(1-\alpha)$ is compared with 1, which is equivalent to comparing ϕ with 1.

The first case when ϕ is less than 1, the long-term economic growth rate is a function of population growth rate, n. The second case, ϕ greater than 1, is growing rapidly. In the third case, when ϕ is equal to 1, then rapid growth (if n is positive) or stable growth (if n is equal to 0).

A special case is $\phi = 1$ and $n = 0$. Then the production function equation 4.3

will look like

$$Y(t) = bK(t), \quad b \equiv B^{1-\alpha} L^{1-\alpha}. \tag{4.5}$$

And the accumulation of capital is determined by the formula

$$\dot{K}(t) = sbK(t) \tag{4.6}$$

The dynamics of this economy is as straightforward as in previous similar cases. K increases stably with velocity sb, this follows from equation (4.6). Thus, the yield also increases with this rate because it is proportional to K. We observe a variant of the model in which long-term growth is endogenous and depends on the rate of savings. It can be added that the model predicts the size of the impact of the savings rate on growth because b is the inverse ratio of capital and production, which is easy to measure.

Since the contribution of capital is greater than its usual contribution, if increased capital increases production due to its direct role in production (term $K(t)^{\alpha}$ in 4.3) and due to indirect promotion of new ideas and thus making all other capital more productive (term $K(t)^{\phi(1-\alpha)}$ in 4.3). Therefore, we can conclude that in this model, the rate of savings affects long-term growth. These models are often called "$Y = AK$" models because the production function is written using the symbol "A" instead of the "b" used in 4.5.

4.1 System Dynamic Model

Now let us take a look at the System Dynamic model of Learning-by-doing (Figure 4.1). We will build this model and investigate the dynamics of the model using software program Stella Architect.

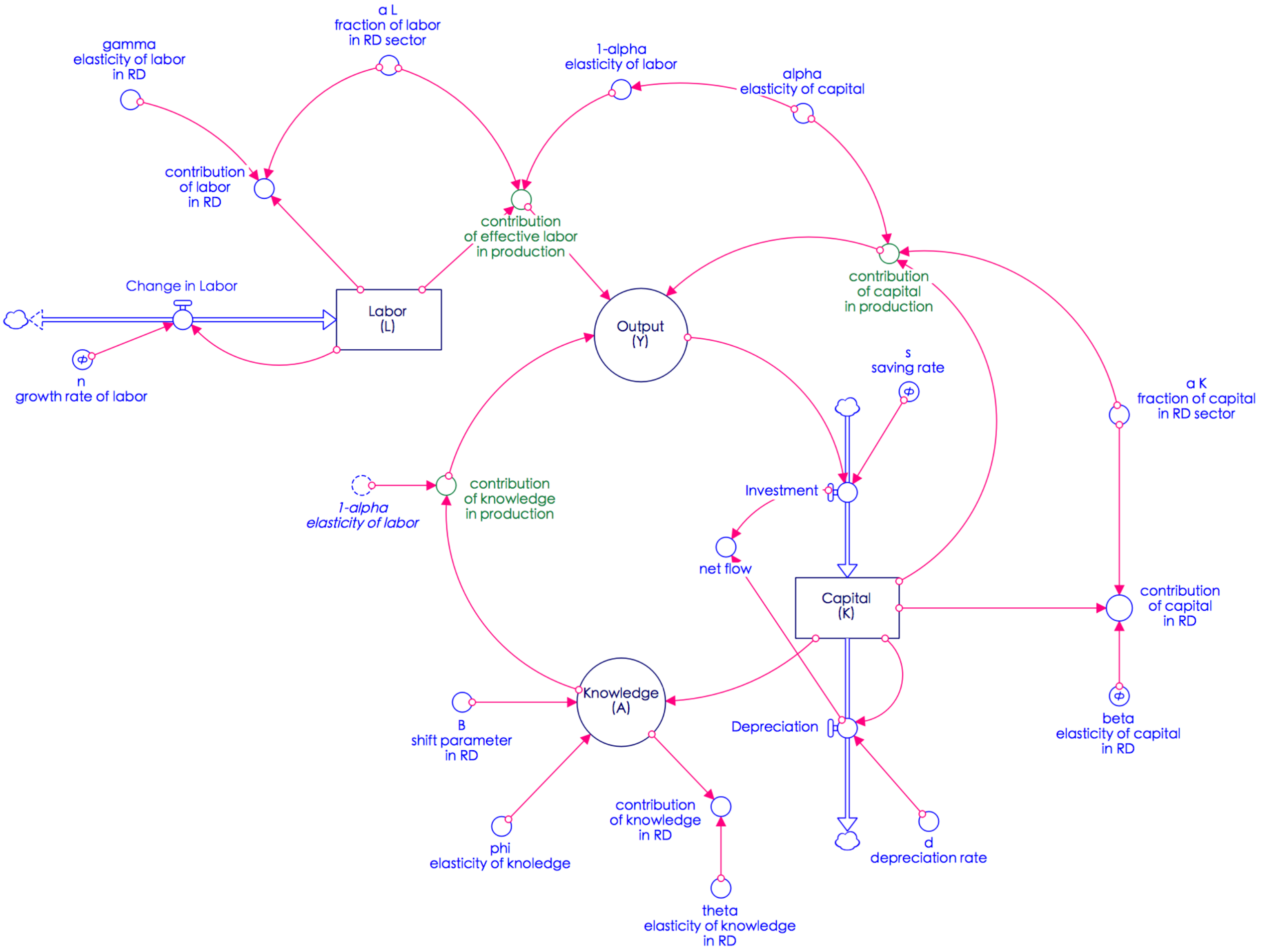

Figure 4.1: The System Dynamic model of Learning-by-doing.

Case 1: $\phi < 1$

We can observe similar behavior when the phi is less than one for the capital of knowledge and output in Figures 4.2 and 4.4. When $\phi = 0,74$, the graph of capital dynamics increases rapidly. And when $\phi = 0,9$, the graphs of knowledge and output are also growing rapidly. But the graphs of capital growth rates, knowledge and output in Figures 4.3 and 4.5 have almost the same behavior. At different values of the parameter n graphs of capital growth rates, knowledge and output first decline and then have a stable behavior.

4.2 Impact of Labor Force Growth Rate

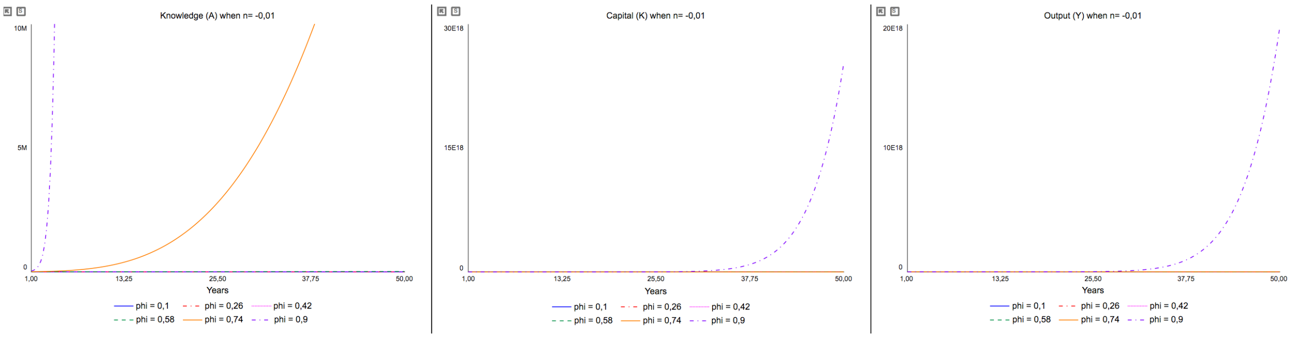

Figure 4.2: The dynamics of knowledge, capital and output when $n = -0,01$ with different value of $\phi < 1$

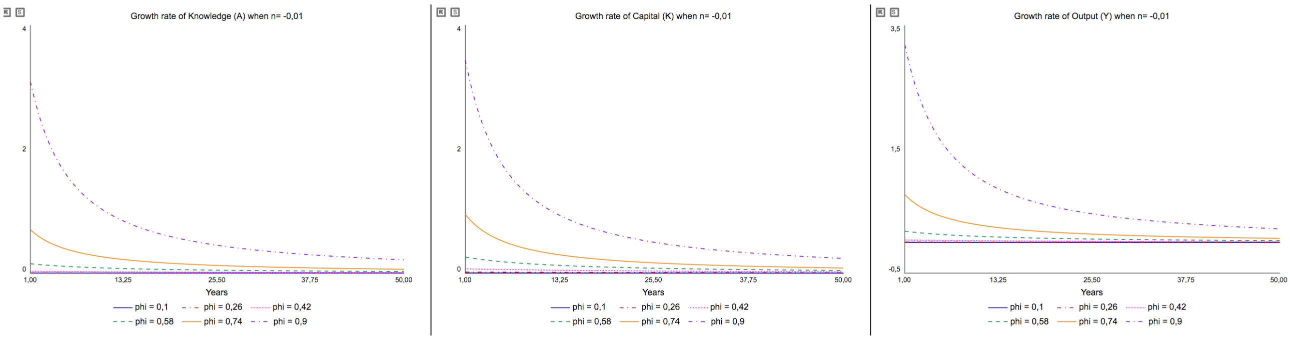

Figure 4.3: The dynamics of growth rates of knowledge, capital and output when $n = -0,01$ with different value of $\phi < 1$

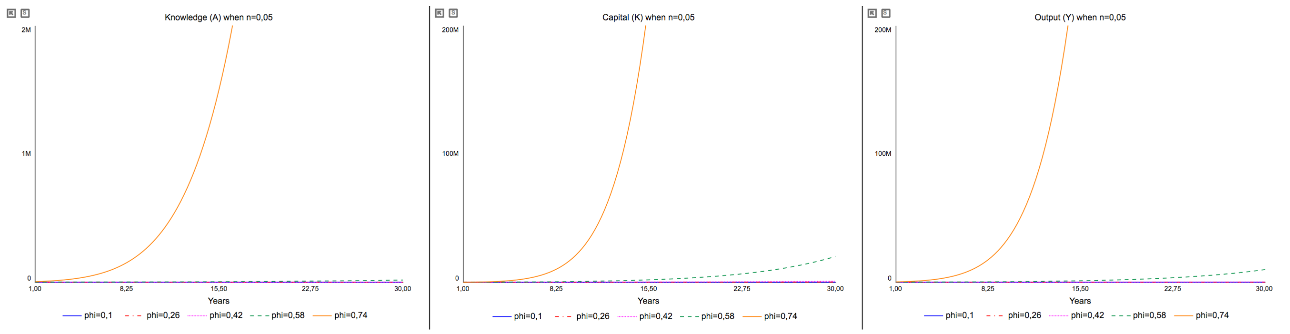

Figure 4.4: The dynamics of knowledge, capital and output when $n = 0,05$ with different value of $\phi < 1$

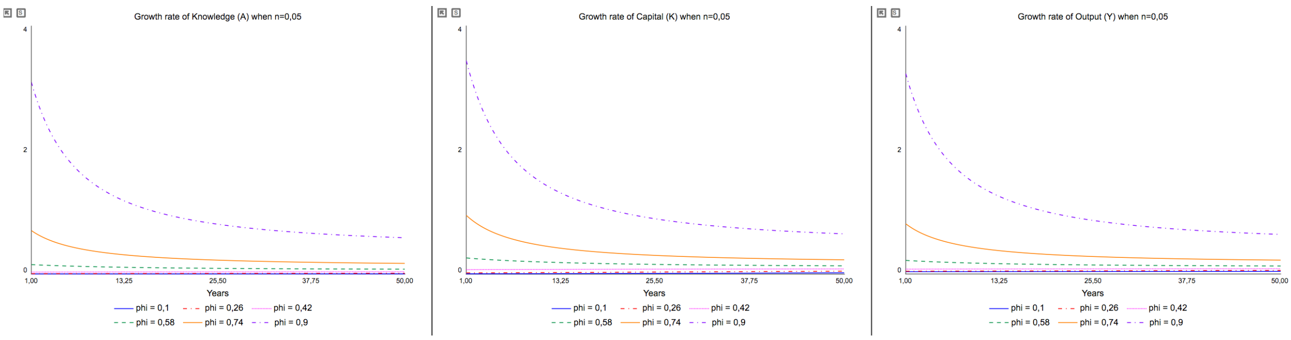

Figure 4.5: The dynamics of growth rates of knowledge, capital and output when $n = 0,05$ with different value of $\phi < 1$

Figures 4.6 and 4.7 show that the graphs are not growing as fast as in the previous figures. And the growth rate depends on the value of the parameter n. The greater n the faster it grows, and the smaller the slower it grows. And in Figure 4.7 starting with $n = 0,06$ graphs of growth rates of knowledge, capital and output begin to decline.

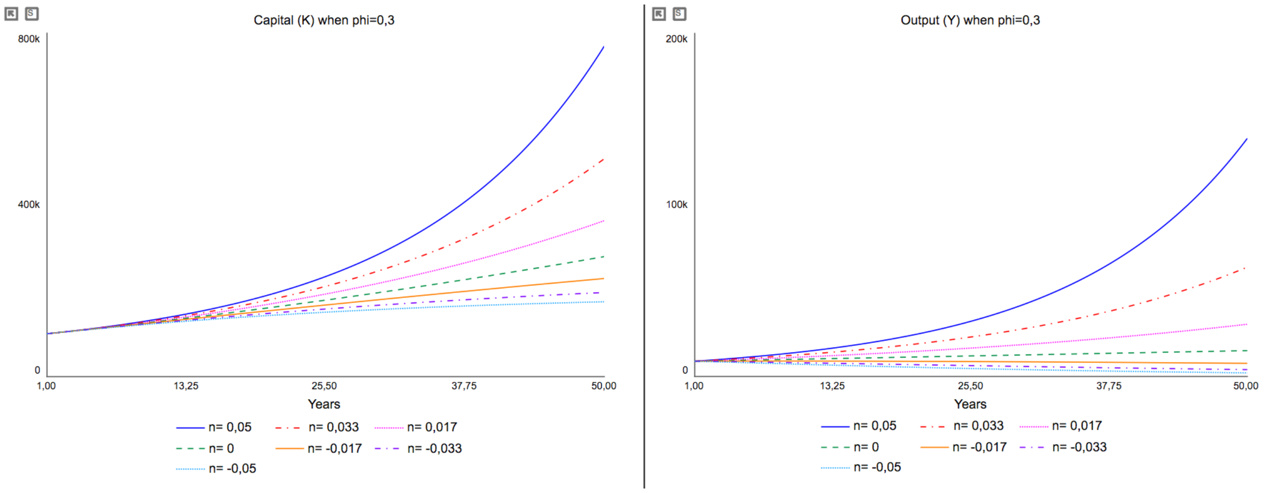

Figure 4.6: The dynamics of knowledge, capital and output when $\phi = 0,3$ with different value of n

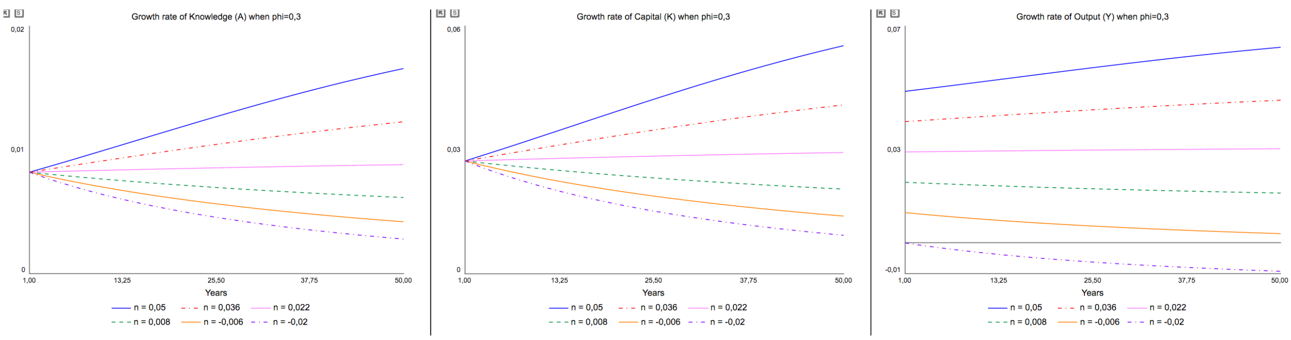

Figure 4.7: The dynamics of growth rates of capital and output when $\phi = 0,3$ with different value of n

Considering the case when $\phi = 0,3$ and n is between -1 and 1, we can conclude that the graphs of the dynamics of knowledge capital and output in Figure 4.8 increase very rapidly when n takes the largest value from this interval, that is 1. And in Figure 4.9 when $n = 1; 0,5$ and $0,2$ graphs of the dynamics of growth rates of knowledge capital and output are growing. For the remaining values of n - stable.

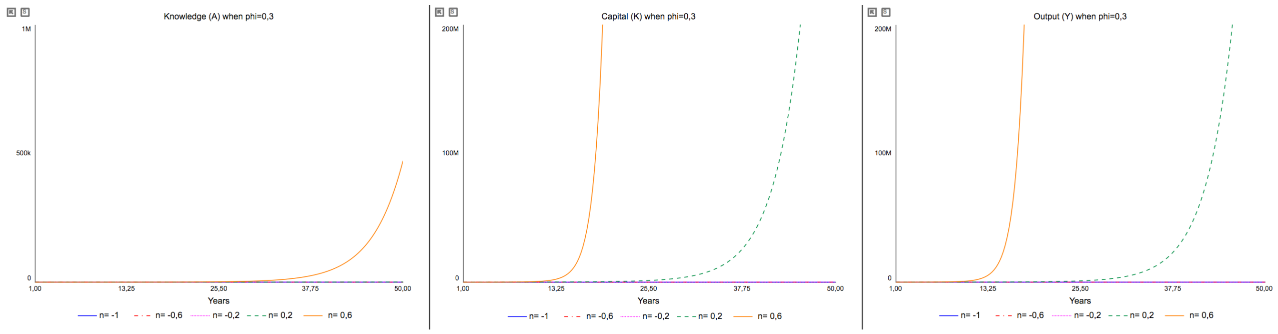

Figure 4.8: The dynamics of knowledge, capital and output when $\phi = 0,3$ and $-1 < n < 1$

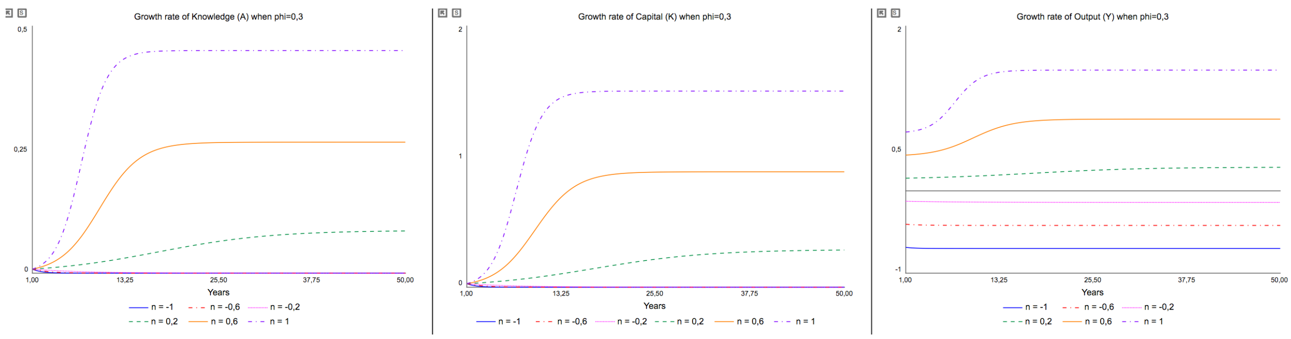

Figure 4.9: The dynamics of growth rates of knowledge, capital and output when $\phi = 0,3$ and $-1 < n < 1$

Now consider in more detail when $\phi = 0,6$ and n is close to zero. The more n the faster the growth. This can be seen in Figure 4.10. And in Figure 4.11 we can see that the growth rate of knowledge of capital and output is declining. And the less n the more they fall.

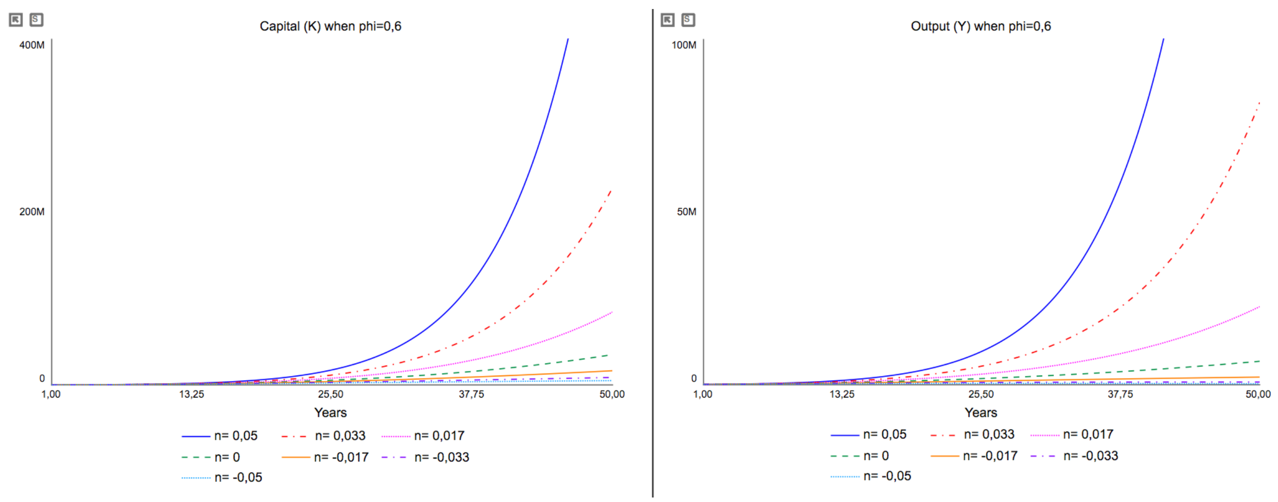

Figure 4.10: The dynamics of capital and output when $\phi = 0,6$ with different value of n

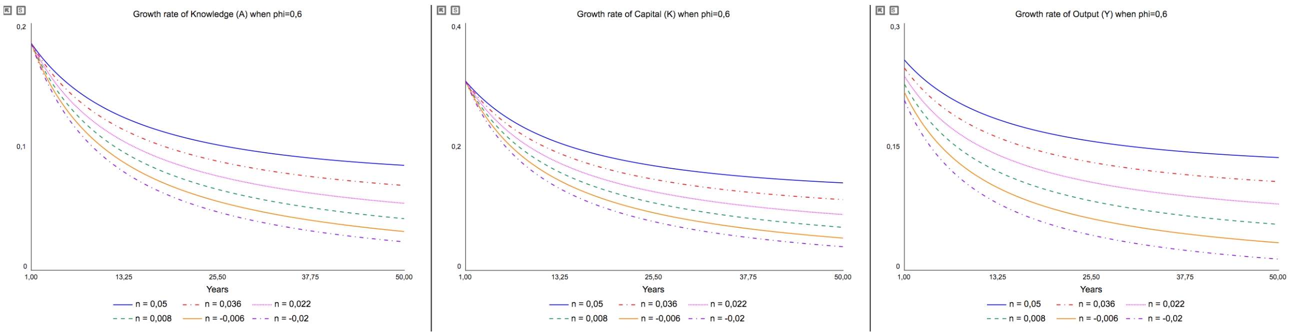

Figure 4.11: The dynamics of growth rates of knowledge, capital and output when $\phi = 0,6$ with different value of n

4.3 Dynamics of Knowledge and Capital Growth Rates for Different Values of Elasticity

Case 2: $\phi > 1$

Figures 4.12-4.16 show that the larger the value of the parameter ϕ, the faster the graphs of the dynamics of knowledge of capital and output, as well as their growth rates.

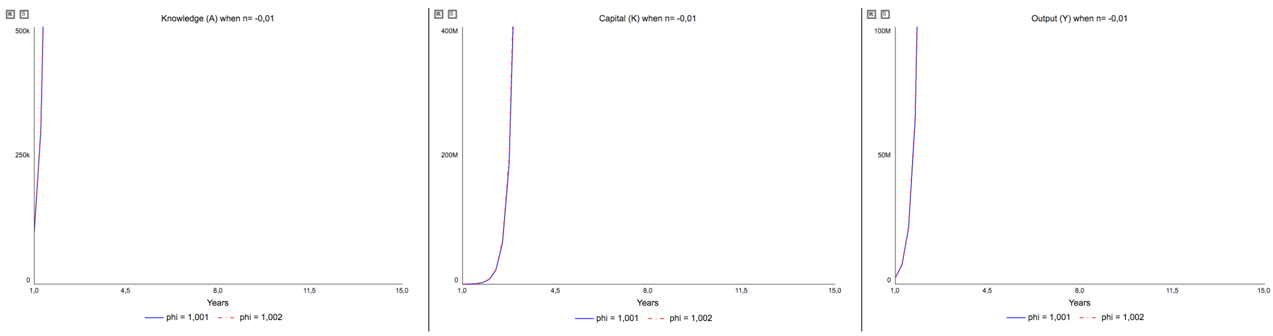

Figure 4.12: The dynamics of knowledge, capital and output when $\phi > 1$ and $n = -0,01$

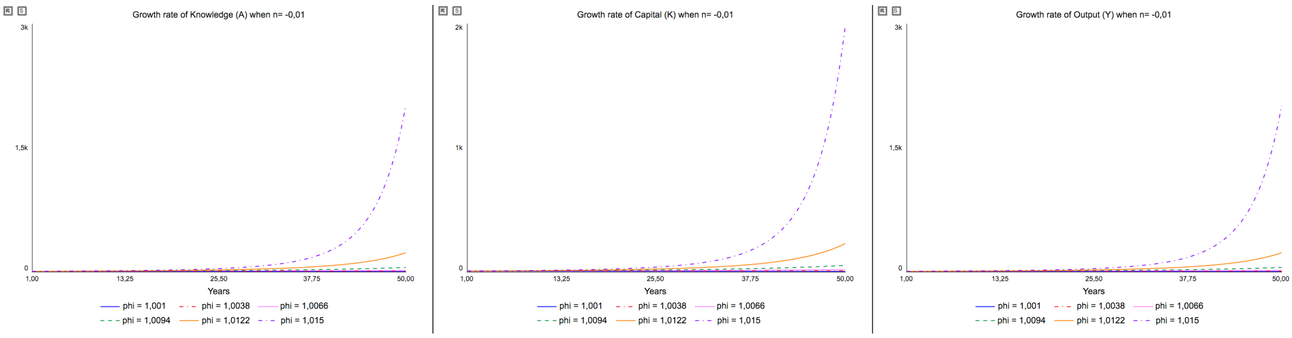

Figure 4.13: The dynamics of growth rates of knowledge, capital and output when when $\phi > 1$ and $n = -0,01$

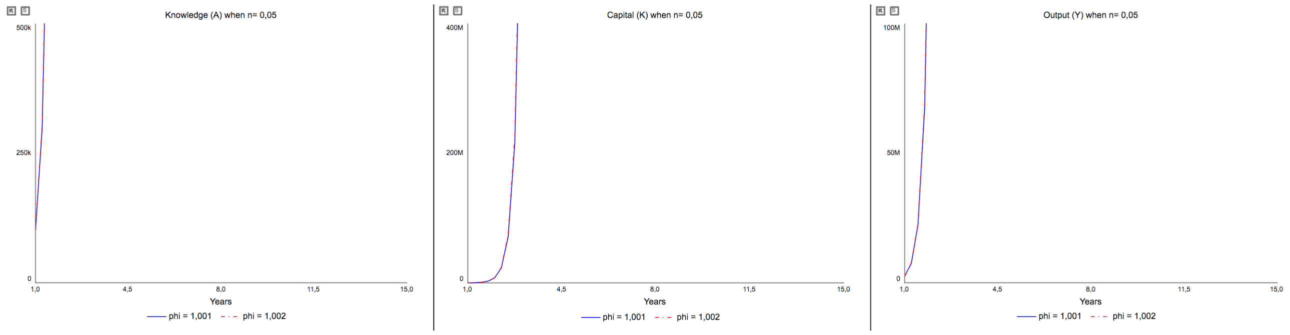

Figure 4.14: The dynamics of knowledge, capital and output when $\phi > 1$ and $n = 0,05$

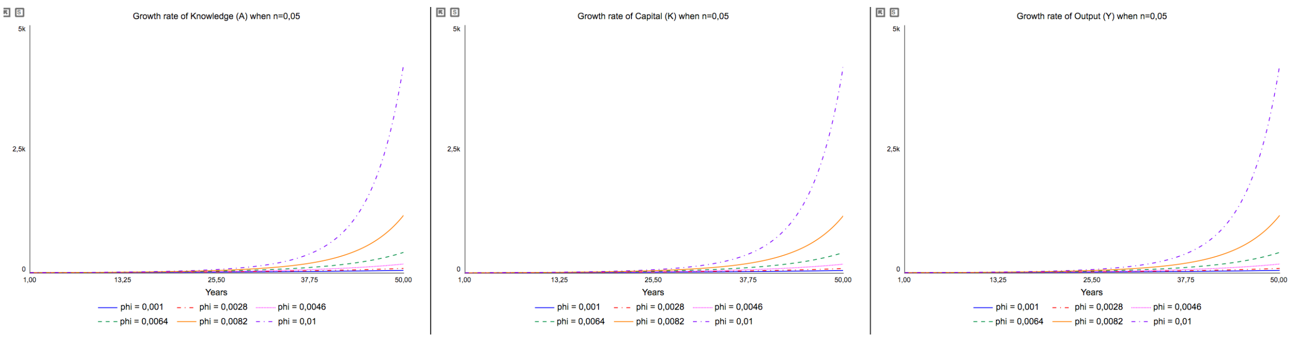

Figure 4.15: The dynamics of growth rates of knowledge, capital and output when $\phi > 1$ and $n = 0,05$

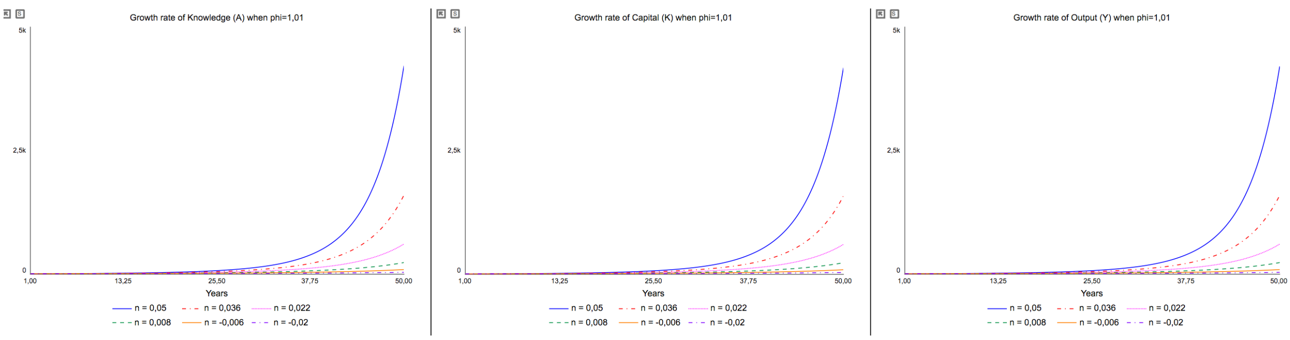

Figure 4.16: The dynamics of growth rates of knowledge, capital and output when $\phi = 1,01$ with different value of n

Case 3: $\phi = 1$

In the third case (ϕ is equal to 1), we can see the rapid growth when n is positive or stable growth when n is equal to 0 in Figure 4.17.

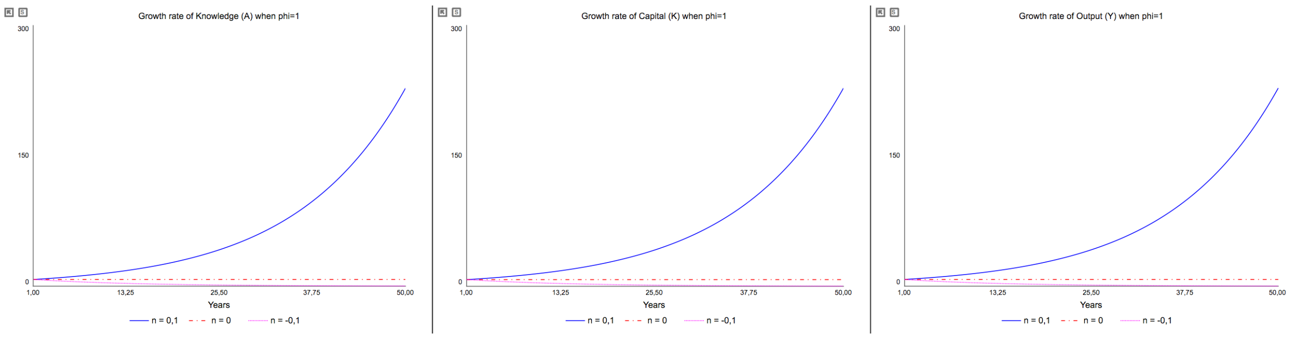

Figure 4.17: The dynamics of growth rates of knowledge, capital and output when $\phi = 1$ with different value of n

Figure 4.18 shows the dynamics of knowledge of capital and output on one chart. First, all variables increase, and then capital and knowledge are stable, and output decreases. Figure 4.19 shows the dynamics of growth rates of knowledge (gA), capital (gK) and output (gY) are also shown in one graph. And here you can see the decline in graphs and then stability.

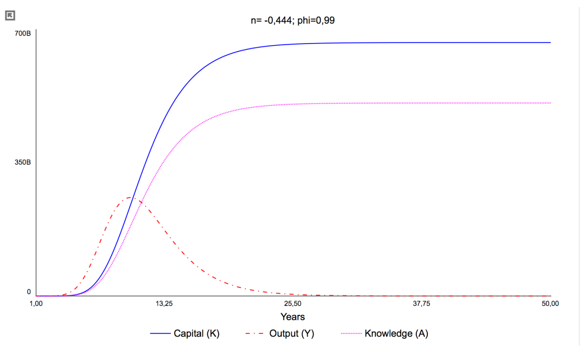

Figure 4.18: The dynamics of knowledge, capital and output when $n = -0,444; \phi = 0,99$

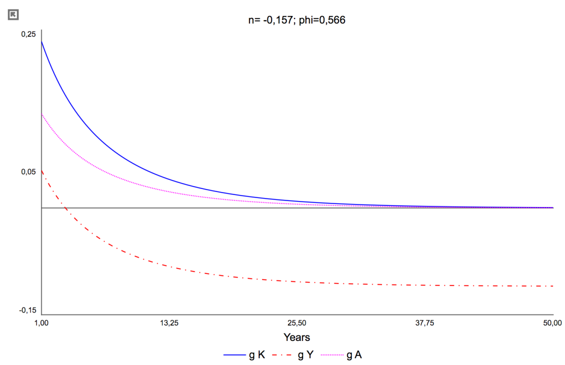

Figure 4.19: The dynamics of growth rates of knowledge, capital and output when $n = -0,157; \phi = 0,566$

Chapter 5

Knowledge as a Side Effect of Goods Production

In this chapter we solve the following problem.

First, let's make the following assumptions:

Output is given by equation 4.1, $Y(t) = K(t)^{\alpha}[A(t)L(t)]^{1-\alpha}$; L is constant and equal to 1; $\dot{K}(t) = sY(t)$ and that knowledge accumulation occurs as a side effect of goods production $\dot{A}(t) = BY(t)$:

1. Find expressions for $g_A(t)$ and $g_K(t)$ in terms of $A(t)$, $K(t)$, and the parameters.

2. Sketch the $\dot{g}_A = 0$ and $\dot{g}_K = 0$ lines in (g_A, g_K) space.

3. Does the economy converge to a balanced growth path? If so, what are the growth rates of K, A, and Y on the balanced growth path?

4. How does an increase in s affect long-run growth?

5.1 Model Specification

The relevant equations are

$$Y(t) = K(t)^{\alpha} A(t)^{1-\alpha} \tag{5.1}$$

$$\dot{K}(t) = sY(t) \tag{5.2}$$

$$\dot{A}(t) = BY(t) \tag{5.3}$$

Now let us take a look at the System Dynamic model of this problem (Figure 5.1). We will build this model and investigate the dynamics of the model using software program Stella Architect.

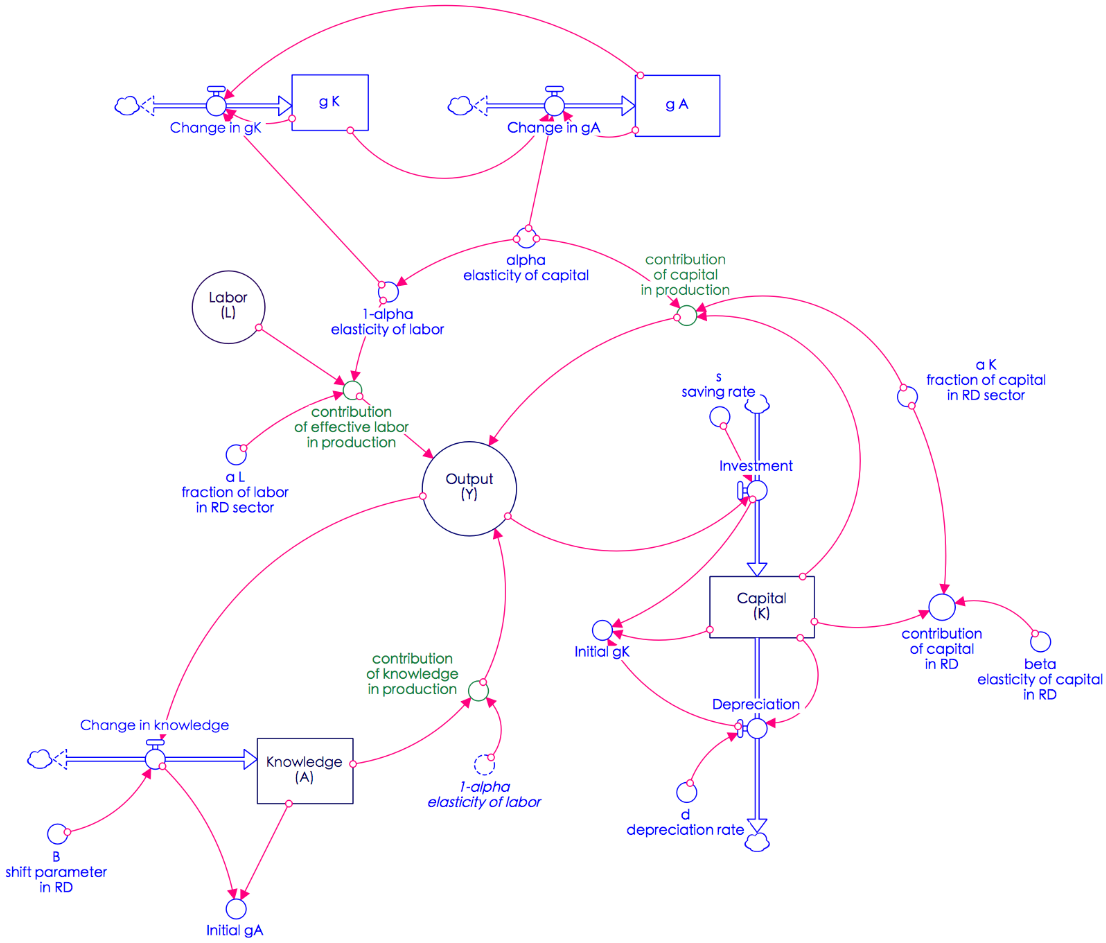

Figure 5.1: The System Dynamic model of problem 1.

1. Substituting equation 5.1 into equation 5.2 yields $\dot{K}(t) = sK(t)^{\alpha}A(t)^{1-\alpha}$. Dividing both sides by $K(t)$ allows us to obtain the following expression for the

growth rate of capital, $g_K(t)$:

$$g_K(t) \equiv \frac{\dot{K}(t)}{K(t)} = sK(t)^{1-\alpha}A(t)^{1-\alpha} \tag{5.4}$$

Substituting equation 5.1 into 5.3 gives us $\dot{A}(t) = BK(t)^{\alpha}A(t)^{1-\alpha}$. Dividing both sides by $A(t)$ allows us to obtain the following expression for the growth rate of knowledge, $g_A(t)$:

$$g_A(t) \equiv \frac{\dot{A}(t)}{A(t)} = BK(t)^{\alpha}A(t)^{-\alpha} \tag{5.5}$$

2. *Capital.* Taking the time derivative of 5.4 yields the following growth rate of the growth rate of capital:

$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)\frac{\dot{K}(t)}{K(t)} + (1-\alpha)\frac{\dot{A}(t)}{A(t)}, \tag{5.6}$$

Or

$$\frac{\dot{g}_K(t)}{g_K(t)} = (1-\alpha)[g_A(t) - g_K(t)]. \tag{5.7}$$

From equation 5.7, g_K will be constant when $g_A = g_K$. Thus the $\dot{g}_K = 0$ locus is a 45° line in (g_A, g_K) space. Also, g_K will be rising when $g_A > g_K$. Thus g_K is rising below the $\dot{g}_K = 0$ line. Lastly, g_K will fall when $g_A < g_K$. Thus g_K is falling above the $\dot{g}_K = 0$ line.

Knowledge. Taking the time derivative of the log of equation 5.5 yields the following growth rate of the growth rate of knowledge:

$$\frac{\dot{g}_A(t)}{g_A(t)} = \alpha\frac{\dot{K}(t)}{K(t)} - \alpha\frac{\dot{A}(t)}{A(t)}, \tag{5.8}$$

Or

$$\frac{\dot{g}_A(t)}{g_A(t)} = \alpha[g_K(t) - g_A(t)]. \tag{5.9}$$

From equation 5.9, g_A will be constant when $g_K = g_A$. Thus the $\dot{g}_A = 0$ locus is also a 45 degree line in (gA , gK) space. Also, g_A will be rising when $g_K > g_A$. Thus above the $\dot{g}_A = 0$ line, g_A will be rising. Finally, g_A will be falling when $g_K < g_A$. Thus below the $\dot{g}_A = 0$ line, g_A will be falling.

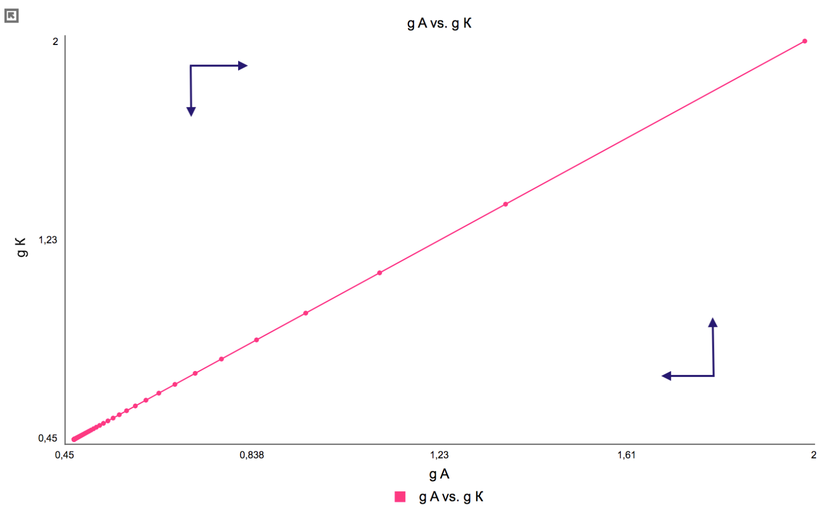

Figure 5.2: The line $g_A = g_k$

5.2 Balancing Path for Capital and Knowledge Growth

3. We can put the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci into one diagram, we can see this in Figure 5.2. Although we can see that the economy will eventually arrive at a situation where $g_K = g_A$ and they are constant, we still do not have enough information to determine the unique balanced growth path. Rewriting equations 5.4 and 5.5 gives us

$$g_K(t) = sK(t)^{\alpha-1}A(t)^{1-\alpha} = s\left[\frac{A(t)}{K(t)}\right]^{1-\alpha}$$

And

$$g_A(t) = BK(t)^{\alpha}A(t)^{-\alpha} = B\left[\frac{A(t)}{K(t)}\right]^{-\alpha}.$$

At any point in time, the growth rates of capital and knowledge are linked because they both depend on the ratio of knowledge to capital at that point in time. It is therefore possible to write one growth rate as a function of the other. From equation 5.5, $\left[\frac{A(t)}{K(t)}\right]^{\alpha} = \frac{B}{g_A(t)}$ or simply

$$\frac{A(t)}{K(t)} = \frac{B}{g_A(t)}^{\frac{1}{\alpha}} \tag{5.1}$$

Substituting equation 5.1 into equation 5.4 gives us

$$g_K(t) = s\left[\frac{B}{g_A(t)}\right]^{\frac{1-\alpha}{\alpha}} \tag{5.2}$$

It must be the case that g_K and g_A lie on the locus satisfying equation 5.2. Note that we are on the $\dot{g}_K = 0$ and $\dot{g}_A = 0$ loci where $g_K = g_A$. Regardless of the initial ratio of A/K the economy starts somewhere on this locus and then moves along it to line $g_A = g_k$. We can see this on Figures 5.3; 5.6 and 5.9. Thus the economy does converge to a unique balanced growth path at line $g_A = g_k$.

The growth rates of knowledge on the balanced growth path, we can observe in Figures 5.4; 5.7 and 5.10. The growth rates of capital on the balanced growth path, we can observe in Figures 5.5; 5.8 and 5.11.

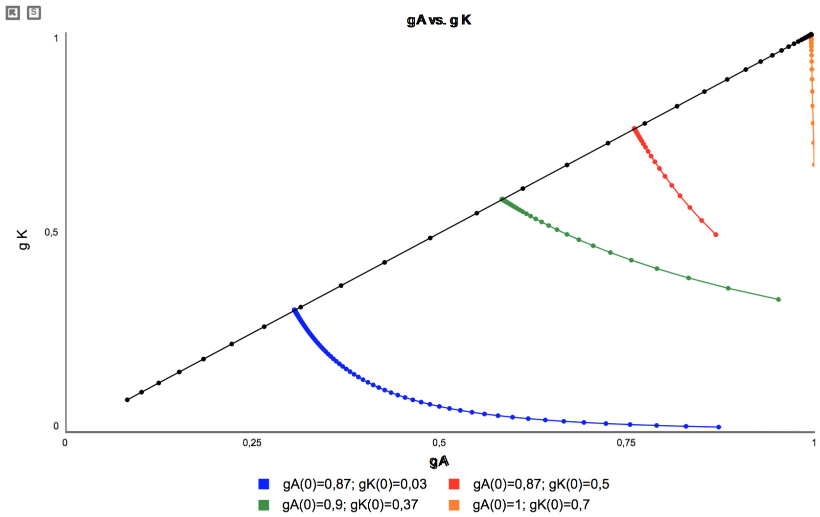

Figure 5.3: Convergence of different points under the line $g_A = g_k$

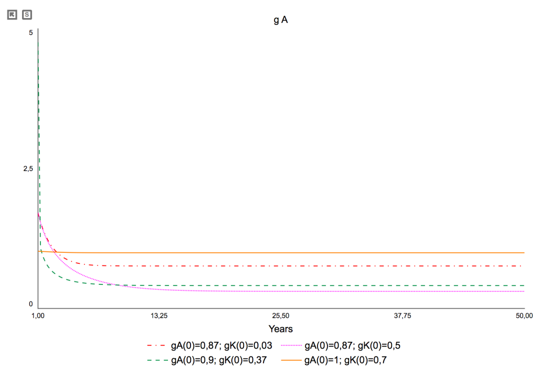

Figure 5.4: Dynamics of the growth rate of knowledge with the convergence of different points under the line $g_A = g_k$

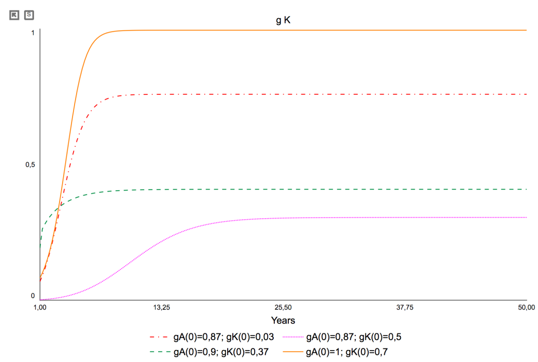

Figure 5.5: Dynamics of the growth rate of capital with the convergence of different points under the line $g_A = g_k$

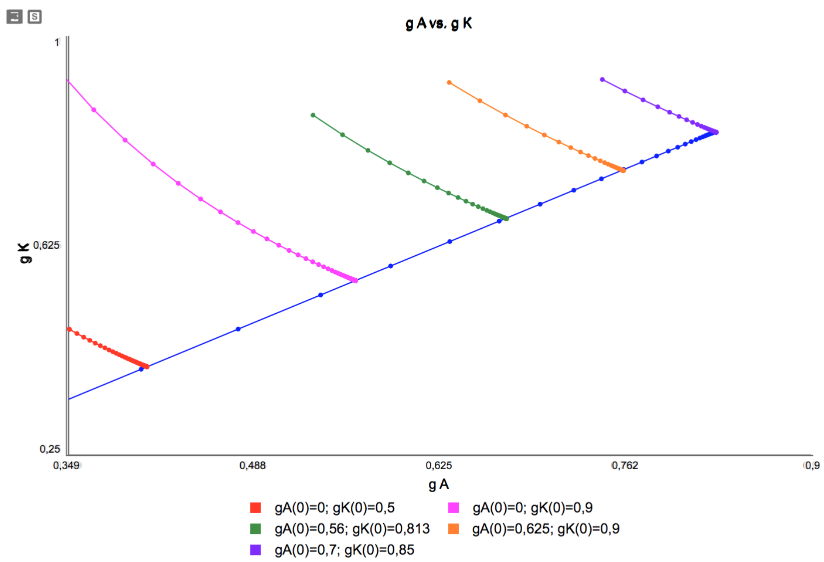

Figure 5.6: Convergence of different points above the line $g_A = g_k$

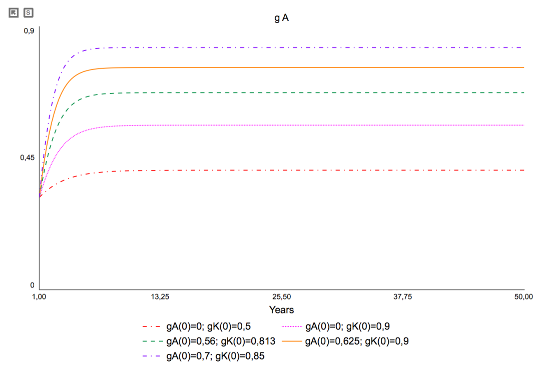

Figure 5.7: Dynamics of the growth rate of knowledge with the convergence of different points above the line $g_A = g_k$

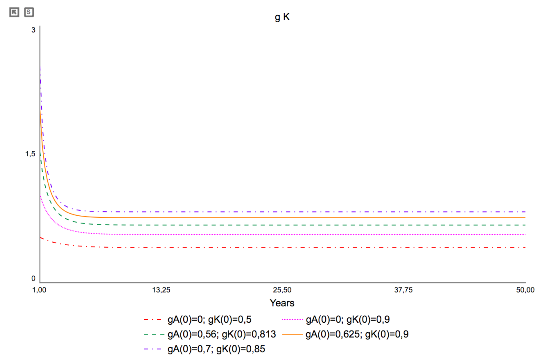

Figure 5.8: Dynamics of the growth rate of capital with the convergence of different points above the line $g_A = g_k$

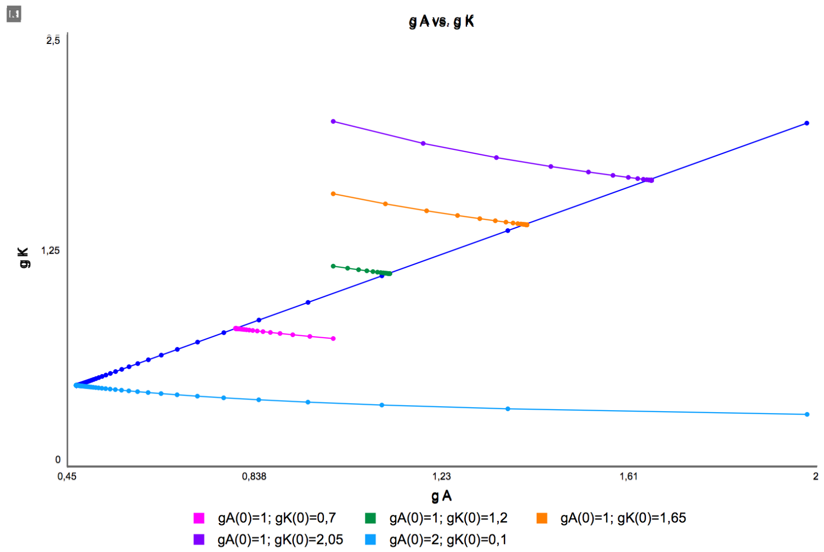

Figure 5.9: Convergence of different points under and above the line $g_A = g_k$

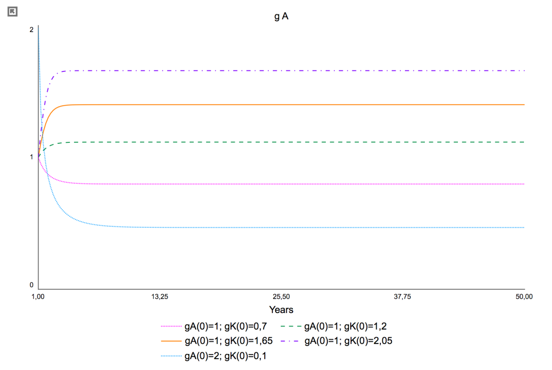

Figure 5.10: Dynamics of the growth rate of knowledge with the convergence of different points under and above the line $g_A = g_k$

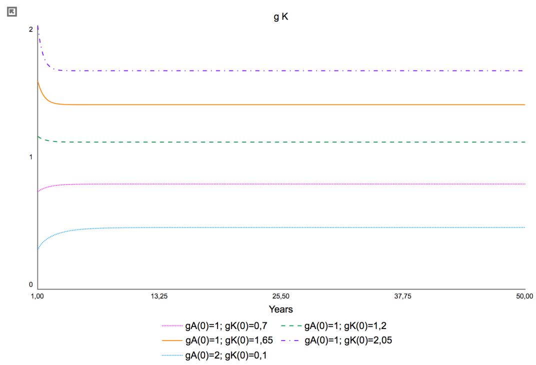

Figure 5.11: Dynamics of the growth rate of capital with the convergence of different points under and above the line $g_A = g_k$

Letting $g*$ denote this common growth rate, then from equation 5.2,

$$g* = s\left[\frac{B}{g*}\right]^{\frac{1-\alpha}{\alpha}}$$

Rearranging to solve for $g*$ yields

$$g* = s^{\alpha}B^{1-\alpha} \tag{5.3}$$

Taking the time derivative of the log of the production function, equation 5.1, yields the growth rate of real output, $\frac{\dot{Y}(t)}{Y(t)} = \alpha g_K(t) + (1-\alpha)g_A(t)$ On the balanced growth path, $g_K = g_A = g*$, and thus

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha g * + (1-\alpha)g* = g* \equiv s^{\alpha}B^{1-\alpha} \tag{5.4}$$

On the balanced growth path, capital, knowledge and output all grow at rate $g*$.

4. Clearly, from equation 5.3, a rise in the saving rate, s, raises g* and thus raises the long- run growth rates of capital, knowledge and output.

   From equations 5.7 and 5.9, neither the $\dot{g}_K = 0$ nor the $\dot{g}_A = 0$ lines shift when s changes since s does not appear in either equation. From equation 5.4, a rise in s causes g_K to jump up. Also, the locus given by equation 5.2 shifts out. So at the moment that s rises, the economy moves from its balanced growth path at point E to a point such as F. It then moves down along the AA locus given by equation 5.2 until it reaches a new balanced growth path at point E_{NEW}.

Chapter 6

Model Based on Firms and Households Decisions

In this chapter we solve the following problem.

Consider the model with equations 4.1 - 4.4. Suppose the output of the firm i's - $Y_i(t) = K_i(t)^{\alpha}[A(t)L_i(t)]^{1-\alpha}$, and that $A(t) = BK(t)$. Where K_i and L_i are the volumes of capital and labor used by firm i, and K is the total stock of capital. Capital and labor earn their private marginal products. The economy is inhabited by infinitely resident households that have the initial capital of the economy. The utility of a representative household takes the form of equations $U = \int_{t=0}^{\infty} e^{-\rho t} u(C(t)) \frac{L(t)}{H} dt$ and $u(C(t)) = \frac{C(t)^{1-\theta}}{1-\theta}, \quad \theta > 0, \quad \rho - n - (1-\theta)g > 0$ of the constant relative deviation to risk. Population growth is zero.

1. (a) What are the private marginal products of capital and labor at firm i as functions of $K_i(t), L_i(t), K(t)$, and the parameters of the model?

   (b) Explain why the capital-labor ratio must be the same at all firms, so

   $K_i(t)/L_i(t) = K(t)/L(t)$ for all i.

   (c) What are $w(t)$ and $r(t)$ as functions of $K(t), L$, and the parameters of the model?

2. What should be the equilibrium growth rate of consumption? Assume for simplicity that the values of the parameters are such that the growth rate is strictly positive and less than the interest rate. Let us outline an explanation of why the equilibrium growth rate of production is equal to the equilibrium growth rate of consumption.

3. Describe how long-run growth is affected by:

   (a) A rise in B.

   (b) A rise in ρ.

   (c) A rise in L.

4. Let's find out whether the equilibrium growth rate is greater than, less than or equal to the socially optimal rate, or it is impossible to say?

6.1 Marginal Product of Capital

1. (a) Substituting the assumption that $A(t) = BK(t)$ into the expression for firm i's output,

$$Y_i(t) = K_i(t)^{\alpha}(A(t)L_i(t))^{1-\alpha}, \text{we get } Y_i(t) = K_i(t)^{\alpha}(B(t)L_i(t))^{1-\alpha}$$

To find the private marginal products of capital and labor, we take the first derivative of output with respect to the firm's choice of capital and labor assuming that the firm takes the aggregate capital stock, K, as given. The private marginal product of capital is therefore

$$\frac{\partial Y_i(t)}{\partial K_i(t)} = \alpha K_i(t)^{\alpha-1}(BK(t)L_i(t))^{1-\alpha} \tag{6.1}$$

or simply

$$\frac{\partial Y_i(t)}{\partial K_i(t)} = \alpha B^{1-\alpha}K(t)^{1-\alpha}\left(\frac{K_i(t)}{L_i(t)}\right)^{-(1-\alpha)} \tag{6.2}$$

The private marginal product of labor is given by

$$\frac{\partial Y_i(t)}{\partial L_i(t)} = (1-\alpha)L_i(t)^{-\alpha}K_i(t)^{\alpha}(BK(t))^{1-\alpha} \tag{6.3}$$

or simply

$$\frac{\partial Y_i(t)}{\partial L_i(t)} = (1-\alpha)B^{1-\alpha}K(t)^{1-\alpha}\left(\frac{K_i(t)}{L_i(t)}\right)^{\alpha} \tag{6.4}$$

6.2 Capital-Labor Ratio

(b) Because factor markets are competitive, at equilibrium the private marginal product of capital and labor cannot differ across firms. We can see from equations 6.2 and 6.4 that this implies the capital-labor ratio will be the same for all firms. Therefore,

$$\frac{K_i(t)}{L_i(t)} = \frac{K(t)}{L(t)}, \quad \text{for all firms.} \tag{6.1}$$

(c) With no depreciation, the real interest rate must equal the private marginal product of capital. From equation 6.2, this implies

$$r(t) = \frac{\partial Y_i(t)}{\partial K_i(t)} = \alpha B^{1-\alpha}K(t)^{1-\alpha}\left(\frac{K_i(t)}{L_i(t)}\right)^{-(1-\alpha)} \tag{6.2}$$

Using the fact that the capital-labor ratio is the same across firms, we can substitute equation 6.1 into equation 6.3 to obtain

$$r(t) = \alpha B^{1-\alpha}K(t)^{1-\alpha}\left(\frac{K(t)}{L}\right)^{-(1-\alpha)}, \tag{6.3}$$

which simplifies to

$$r(t) = \alpha B^{1-\alpha}L^{1-\alpha} = \alpha b, \tag{6.4}$$

where $b \equiv B^{1-\alpha}L^{1-\alpha}$ With no population growth, L is constant, and thus so is the real interest rate. The real wage must equal the marginal product of labor. From equation 6.4 this implies

$$w(t) = \frac{\partial Y_i(t)}{\partial L_i(t)} = (1-\alpha)B^{1-\alpha}K(t)^{1-\alpha}.\frac{K_i(t)}{L_i(t)}^{\alpha} \tag{6.5}$$

Again, using the fact that the capital-labor ratio is the same across firms, we can substitute equation 6.1 into equation 6.5 to obtain

$$w(t) = (1-\alpha)B^{1-\alpha}K(t)^{1-\alpha}\frac{K(t)}{L}^{\alpha}, \tag{6.6}$$

which simplifies to

$$w(t) = (1-\alpha)B^{1-\alpha}K(t)L^{-\alpha} = (1-\alpha)B^{1-\alpha}L^{1-\alpha}\frac{K(t)}{L}, \tag{6.7}$$

or simply

$$w(t) = (1-\alpha)b\frac{K(t)}{L}. \tag{6.8}$$

6.3 Equilibrium Growth Rate of Consumption

2. Using the hint, since utility of the representative household takes the constant-relative-risk-aversion- form, consumption growth in equilibrium will be

$$\frac{\dot{C}(t)}{C(t)} = \frac{r(t)-\rho}{\theta}. \tag{6.1}$$

Substituting equation 2 for the real interest rate into equation 6.1 yields

$$\frac{\dot{C}(t)}{C(t)} = \frac{\alpha b-\rho}{\theta}, \tag{6.2}$$

where $b \equiv BL$. Note that with no population growth so that L is constant, consumption growth is constant as well. Using the zero-profit condition, we can write output as

$$Y(t) = r(t)K(t) + w(t)L. \tag{6.3}$$

Substituting equations and into equation gives us

$$Y(t) = \alpha bK(t) + (1-\alpha)b\left(\frac{K(t)}{L}\right)L, \tag{6.4}$$

which simplifies to

$$Y(t) = bK(t). \tag{6.5}$$

Since b is a constant then output and capital grow at the same rate. Capital accumulation is then given by

$$\dot{K}(t) = sbK(t), \tag{6.6}$$

where s is the saving rate. Thus, the growth rate of the capital stock is given by

$$\frac{\dot{K}(t)}{K(t)} = sb, \tag{6.7}$$

and so the growth rate of output also equals sb. Since $C = (1\backslash s)Y$, we can write the saving rate as

$$s = 1 - \frac{C}{Y}. \tag{6.8}$$

Thus, we can write the growth rate of output as

$$\frac{\dot{Y}(t)}{Y(t)} = b\left(1 - \frac{C(t)}{Y(t)}\right). \tag{6.9}$$

If output growth were less than consumption growth, C/Y would rise over time. Output growth and capital growth would turn negative, which is not an allowable path. If output growth were greater than consumption growth, C/Y would fall to 0 over time. Output growth and capital growth would approach b. This implies that growth would eventually exceed the real interest rate, which is αb, and so this is also not an allowable path. Thus, the equilibrium growth rates of output and consumption must be equal.

6.4 Long-run Growth

3. (a) We can take the derivative of the growth rate of output (which equals the growth rate of consumption) with respect to B to obtain

$$\frac{\partial\left[\frac{\dot{Y}(t)}{Y(t)}\right]}{\partial B} = \frac{\partial\left[\frac{\alpha B^{1-\alpha}L^{1-\alpha}-\rho}{\theta}\right]}{\partial B} = \frac{\alpha(1-\alpha)L^{1-\alpha}}{\theta B^{\alpha}} > 0.$$

Thus an increase in B increases long-run growth.

(b) We can take the derivative of the growth rate of output with respect to ρ to obtain

$$\frac{\partial\left[\frac{\dot{Y}(t)}{Y(t)}\right]}{\partial\rho}=\frac{\partial\left[\frac{\alpha B^{1-\alpha}L^{1-\alpha}-\rho}{\theta}\right]}{\partial\rho}=-\frac{1}{\theta}<0.$$

Thus an increase in´decreases long-run growth.

(c) We can take the derivative of the growth rate of output with respect to L to obtain

$$\frac{\partial\left[\frac{\dot{Y}(t)}{Y(t)}\right]}{\partial L}=\frac{\partial\left[\frac{\alpha B^{1-\alpha}L^{1-\alpha}-\rho}{\theta}\right]}{\partial L}=\frac{\alpha(1-\alpha)B^{1-\alpha}}{\theta L^{\alpha}}>0.$$

Thus an increase in L increases long-run growth.

4. The equilibrium growth rate is less than the socially optimal growth rate. A social planner would internalize the knowledge spillovers and would set the growth rate of consumption dependent on the social return to capital, not the private return. We know that the private marginal product of capital is αb and the social marginal product is b (returns to capital are constant at the social level). Therefore, unless $\alpha = 1$, the growth rate set by the social planner would be greater than the decentralized equilibrium growth rate.

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