Macro Topics: Introduction to Matlab

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Lecture notes (December 23)

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Topics Covered Today

Linear-Quadratic Problem

- Linear-Quadratic Problem
- Discounted Linear-Quadratic Problem
- Stochastic Discounted Linear-Quadratic Problem
- LQ-ization
- LQ Problem after LQ-ization

This lecture is based on Ljungqvist & Sargent book, Chapter 5.

Linear-Quadratic Problem I

The problem is written in a special form, with quadratic objective function and linear constraint. In the deterministic case, we have

$$\max \sum_{t=0}^{\infty} x_t^T R x_t + u_t^T Q u_t,$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad x_0 \text{ given.}$$

- Here x_t is a vector of state variables, u_t is a vector of control variables.
- Assumptions on the matrices: R is negative semidefinite symmetric matrix, Q is negative definite symmetric matrix. "Semi" means some eigenvalues of R are allowed to be zero.
- Both $x_t^T R x_t$ and $u_t^T Q u_t$ are quadratic forms and thus are scalars.
- Solution method: guess and verify.

Linear-Quadratic Problem II

Our guess is: $V(x) = x_t^T P x_t$, where P is unknown symmetric matrix.

$$V(x) = x^T P x = \max_u \left\{ x^T R x + u^T Q u + (Ax + Bu)^T P (Ax + Bu) \right\}.$$

To take derivatives on the RHS, we need expressions for derivatives of quadratic forms:

$$\frac{\partial}{\partial u_i} u^T Q u = \frac{\partial}{\partial u_i} \sum_{j,k} u_j q_{jk} u_k = \sum_{j,k} (\delta_{ij} q_{jk} u_k + u_j q_{jk} \delta_{ik}),$$

where

$$\delta_{ik} = 1 \quad \text{if} \quad i = k,$$

$$\delta_{ik} = 0 \quad \text{otherwise.}$$

Linear-Quadratic Problem III

Summation over j in the first term and over k in the second gives

$$\sum_k q_{ik} u_k + \sum_j q_{ji} u_j,$$

and the first sum is clearly seen to be just i^{th} element of the vector Qu. For the second term, notice that q_{ji} is (i, j) element of the matrix Q^T, and thus the second term becomes $\sum_j q_{ij}^T u_j$ which is i^{th} element of vector $Q^T u$. Collecting all the terms together, we get

$$\frac{\partial}{\partial u} u^T Q u = \left(Q + Q^T \right) u$$

if we want to write derivatives as column vectors. For a symmetric matrix Q, the derivative is just $2Qu$.

Linear-Quadratic Problem IV

Two other derivatives we will need are

$$\frac{\partial}{\partial u_i} u^T R y \quad \text{and} \quad \frac{\partial}{\partial u_i} y^T R u.$$

To get the first one, again write

$$\frac{\partial}{\partial u_i} u^T R y = \frac{\partial}{\partial u_i} \sum_{j,k} u_j r_{jk} y_k = \sum_{j,k} \delta_{ij} r_{jk} y_k = \sum_{k} r_{ik} y_k,$$

or i^{th} element of the vector Ry. Therefore, the derivative of $u^T Ry$ with respect to u equals Ry.

Similarly,

$$\frac{\partial}{\partial u_i} y^T R u = \frac{\partial}{\partial u_i} \sum_{j,k} y_j r_{jk} u_k = \sum_{j,k} y_j r_{jk} \delta_{ik} = \sum_{j} r_{ij}^T y_j,$$

which is i^{th} element of vector $R^T y$. Therefore, the derivative of $y^T Ru$ with respect to u equals $R^T y$.

Linear-Quadratic Problem V

Given these expressions for the derivatives, write the FOC as

$$\frac{\partial}{\partial u}\left\{x^T R x + u^T Q u + (Ax+Bu)^T P (Ax+Bu)\right\} = 0,$$

$$\frac{\partial}{\partial u}\left\{u^T Q u + x^T A^T P B u + u^T B^T P A x + u^T B^T P B u\right\} = 0,$$

$$\left(Q + Q^T\right) u + \left(A^T P B\right)^T x + B^T P A x + \left(B^T P B + B^T P B\right) u = 0,$$

$$2Qu + 2B^T P B u + 2B^T P A x = 0,$$

where we used the fact that Q and P are symmetric matrices. Solving the last line for u, we get

$$u = -\left(Q + B^T P B\right)^{-1} B^T P A x.$$

If we know matrix P, the problem is solved: a vector of controls is a linear function of the vector of states. However, P is unknown.

Linear-Quadratic Problem VI

Plug in the optimal u into the Bellman equation. On the right hand side, we have

$$x^T Rx + u^T Qu + x^T A^T PAx + x^T A^T PBu + u^T B^T PAx + u^T B^T PBu =$$

$$= x^T Rx + x^T A^T PAx + x^T A^T PBu + u^T \underbrace{\left(Qu + B^T PBu + B^T PAx\right)}_{=0 \text{ by the FOC}},$$

$$x^T Px = x^T Rx + x^T A^T PAx - x^T A^T PB \left(Q + B^T PB\right)^{-1} B^T PAx,$$

$$P = R + A^T PA - A^T PB \left(Q + B^T PB\right)^{-1} B^T PA.$$

The last line is a Riccati matrix equation. There are methods of solving quadratic matrix equations. Assuming that we have solved for P, we can then plug it into the expression for u and get the policy function.

Linear-Quadratic Problem VI

In reality, we do not need to rely on subroutines solving Riccati equations. We notice that $x^T Px$ is the value function, and should be obtained as a limit of value function iterations. Given that all value function iterations are also quadratic forms, we could iterate on the Riccati equation:

$$P_{j+1} = R + A^T P_j A - A^T P_j B \left(Q + B^T P_j B \right)^{-1} B^T P_j A,$$

starting from any bounded P_j such as $P \equiv 0$. Given that Q is negatively definite (and thus invertible), these iterations produce valid matrices at every step.

Discounted Linear-Quadratic Problem I

In economics our problems are usually discounted:

$$\max \sum_{t=0}^{\infty} \beta^t \left(x_t^T R x_t + u_t^T Q u_t \right),$$

$$\text{s.t.} \quad x_{t+1} = A x_t + B u_t, \quad x_0 \text{ given.}$$

Again, the guess is: $V(x) = x_t^T P x_t$.

$$V(x) = x^T P x = \max_u \left\{ x^T R x + u^T Q u + \beta (Ax + Bu)^T P (Ax + Bu) \right\}.$$

Discounted Linear-Quadratic Problem II

Taking derivatives, we get

$$\frac{\partial}{\partial u}\left\{x^T Rx + u^T Qu + \beta(Ax+Bu)^T P(Ax+Bu)\right\} = 0,$$

$$\frac{\partial}{\partial u}\left\{u^T Qu + \beta x^T A^T PBu + \beta u^T B^T PAx + \beta u^T B^T PBu\right\} = 0,$$

$$\left(Q + Q^T\right) u + \beta\left(A^T PB\right)^T x + \beta B^T PAx + \beta\left(B^T PB + B^T PB\right) u = 0,$$

$$2Qu + 2\beta B^T PBu + 2\beta B^T PAx = 0,$$

where we used the fact that Q and P are symmetric matrices. Solving the last line for u, we get

$$u = -\left(Q + B^T \beta PB\right)^{-1} B^T \beta PAx.$$

Discounted Linear-Quadratic Problem III

Plug in the optimal u into the Bellman equation. On the right hand side, we have

$$x^T Rx + u^T Qu + \beta x^T A^T PAx + \beta x^T A^T PBu + \beta u^T B^T PAx + \beta u^T B^T PBu =$$

$$= x^T Rx + \beta x^T A^T PAx + \beta x^T A^T PBu + u^T \underbrace{\left(Qu + \beta B^T PBu + \beta B^T PAx\right)}_{=0 \text{ by the FOC}},$$

$$x^T Px = x^T Rx + x^T A^T \beta PAx - x^T A^T \beta PB \left(Q + B^T \beta PB\right)^{-1} B^T \beta PAx,$$

$$P = R + A^T \beta PA - A^T \beta PB \left(Q + B^T \beta PB\right)^{-1} B^T \beta PA.$$

Discounted Linear-Quadratic Problem IV

In other words, the only difference with the formula for undiscounted case is that in every place where we used to have P we have βP now.

Once again, we could iterate on the Riccati equation:

$$P_{j+1} = R + A^T \beta P_j A - A^T \beta P_j B \left(Q + B^T \beta P_j B \right)^{-1} B^T \beta P_j A,$$

starting from any bounded P_j such as $P \equiv 0$.

Stochastic Discounted Linear-Quadratic Problem I

Now assume that we have the following problem:

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left(x_t^T R x_t + u_t^T Q u_t \right),$$

$$\text{s.t.} \quad x_{t+1} = Ax_t + Bu_t + \epsilon_{t+1}, \quad x_0 \text{ given},$$

$$\mathbb{E}[\epsilon_t] = 0, \quad \mathbb{E}\left[\epsilon_t \epsilon_t^T\right] = \Sigma.$$

Make a guess:

$$V(x) = x^T P x + d,$$

write the Bellman equation

$$x^T P x + d \; =$$

$$= \max_u \left\{ x^T R x + u^T Q u + \beta \mathbb{E}\left[(Ax + Bu + \epsilon)^T P (Ax + Bu + \epsilon) + d \right] \right\}.$$

Stochastic Discounted Linear-Quadratic Problem II

When we take the FOC, all the extra terms with both u and ϵ will cancel out as ϵ enters in the first power and is zero in expectation.

Therefore, the policy function remains the same! This is a manifestation of the *certainty equivalence* principle: with quadratic objective function, linear constraints, and additive noise, the problem could be solved under perfect foresight, when all the variables are taken to equal their averages.

There is only one difference: the extra constant in the value function. To get it, write the terms involving constants in the Bellman equation separately:

$$d = \beta d + \beta \mathbb{E}\left[\epsilon_t^T P \epsilon_t\right].$$

$\epsilon_t^T P \epsilon_t$ is a scalar, and every scalar equals its own trace,

$$\epsilon_t^T P \epsilon_t = trace\left(\epsilon_t^T P \epsilon_t\right).$$

Stochastic Discounted Linear-Quadratic Problem III

For the trace, we have a convenient property

$$trace(ABC) = trace(CAB),$$

and thus

$$trace\left(\epsilon_t^T P \epsilon_t\right) = trace\left(\epsilon_t \epsilon_t^T P\right).$$

Given that the trace is a linear operation, it is interchangable with the expectations operator:

$$\beta \mathbb{E}\left[\epsilon_t^T P \epsilon_t\right] = \beta \mathbb{E}\left[trace\left(\epsilon_t^T P \epsilon_t\right)\right] = \beta \mathbb{E}\left[trace\left(\epsilon_t \epsilon_t^T P\right)\right] =$$

$$= \beta trace\left(\mathbb{E}\left[\epsilon_t \epsilon_t^T P\right]\right) = \beta trace\left(\mathbb{E}\left[\epsilon_t \epsilon_t^T\right] P\right) = \beta trace(\Sigma P).$$

Therefore, the constant in the value function is given as

$$d = \frac{\beta}{1-\beta} trace(\Sigma P).$$

LQ-ization I

It is very nice when the objective function is exactly quadratic. However, if you attempt to perform a Taylor expansion of any objective function up to the second order, there are likely to be constants and linear terms. What do we do in this case?

Suppose our problem is

$$\max \sum_{t=0}^{\infty} \beta^t F(x_t, u_t),$$

$$\text{s.t.} \quad x_{t+1} = Ax_t + Bu_t, \quad x_0 \text{ given.}$$

LQ-ization II

Expanding the period return function up to the second order, we get $F(x, u)$ approximately equal to

$$F(\bar{x}, \bar{u}) + \left(F_x(\bar{x}, \bar{u})^T, F_u(\bar{x}, \bar{u})^T\right) \begin{bmatrix} x - \bar{x} \\ u - \bar{u} \end{bmatrix} +$$

$$+ \frac{1}{2} \left((x - \bar{x})^T, (u - \bar{u})^T\right) \begin{bmatrix} F_{xx}(\bar{x}, \bar{u}) & F_{xu}(\bar{x}, \bar{u}) \\ F_{ux}(\bar{x}, \bar{u}) & F_{uu}(\bar{x}, \bar{u}) \end{bmatrix} \begin{bmatrix} x - \bar{x} \\ u - \bar{u} \end{bmatrix}.$$

How to get rid of constants and linear terms here?

LQ-ization III

Introduce a vector z_t as

$$z_t = \begin{bmatrix} 1 \\ x_t \\ u_t \end{bmatrix},$$

with the stationary value

$$\bar{z} = \begin{bmatrix} 1 \\ \bar{x} \\ \bar{u} \end{bmatrix}.$$

Then, define the matrix M,

$$M = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix}$$

so that $z_t^T M z_t$ equals the 2nd order Taylor expansion (we also want M to be symmetric).

LQ-ization IV

Is it possible? It turns out that if

$$m_{11} = F(\bar{x},\bar{u}) - \bar{x}^T F_x(\bar{x},\bar{u}) - \bar{u}^T F_u(\bar{x},\bar{u}) +$$

$$+ \frac{1}{2}\bar{x}^T F_{xx}(\bar{x},\bar{u})\bar{x} + \bar{x}^T F_{xu}(\bar{x},\bar{u})\bar{u} + \frac{1}{2}\bar{u}^T F_{uu}(\bar{x},\bar{u})\bar{u},$$

$$m_{12} = m_{21}^T = \frac{1}{2}\left(F_x(\bar{x},\bar{u})^T - \bar{x}^T F_{xx}(\bar{x},\bar{u}) - \bar{u}^T F_{ux}(\bar{x},\bar{u})\right),$$

$$m_{13} = m_{31}^T = \frac{1}{2}\left(F_u(\bar{x},\bar{u})^T - \bar{x}^T F_{xu}(\bar{x},\bar{u}) - \bar{u}^T F_{uu}(\bar{x},\bar{u})\right),$$

and the other three matrices are just

$$m_{22} = \frac{1}{2}F_{xx}(\bar{x},\bar{u}),$$

$$m_{23} = m_{32}^T = \frac{1}{2}F_{xu}(\bar{x},\bar{u}),$$

$$m_{33} = \frac{1}{2}F_{uu}(\bar{x},\bar{u}),$$

then we get exactly what we wanted.

LQ-ization V

How did we get this? Basically, this is the method of undetermined coefficients once again: on the (very long) LHS we have approximate 2nd order Taylor expansion, and on the RHS we have $z_t^T M z_t$ written componentwise. Then, we want to make sure that similar functions have similar coefficients. We need to remember that $F_{xu} = F_{ux}^T$.

The full three by three case is rather messy, but let say we have only x. In this case,

$$\left(1, x^T\right) \begin{bmatrix} m_{11} & m_{12} \\ m_{12}^T & m_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x \end{bmatrix} = m_{11} + m_{12}x + (m_{12}x)^T + x^T m_{22} x,$$

and this should be equal to

$$\bar{F} + F_x^T x - F_x^T \bar{x} + \frac{1}{2} x^T F_{xx} x + \frac{1}{2} \bar{x}^T F_{xx} \bar{x} - \frac{1}{2} \bar{x}^T F_{xx} x - \frac{1}{2} x^T F_{xx} \bar{x}.$$

LQ-ization VI

Equating quadratic forms $x^T m_{22} x$ and $\frac{1}{2} x^T F_{xx} x$ immediately gives that

$$m_{22} = \frac{1}{2} F_{xx}.$$

Next, we write

$$m_{12}x + (m_{12}x)^T = F_x^T x - \frac{1}{2}\bar{x}^T F_{xx} x - \frac{1}{2} x^T F_{xx} \bar{x} \ =$$

$$= \ \frac{1}{2} F_x^T x + \frac{1}{2}\left(F_x^T x\right)^T - \frac{1}{2}\bar{x}^T F_{xx} x - \frac{1}{2} x^T F_{xx}\bar{x} \ =$$

$$= \ \frac{1}{2}\left(F_x^T - \bar{x}^T F_{xx}\right) x + \frac{1}{2}\left[\left(F_x^T - \bar{x}^T F_{xx}\right) x\right]^T,$$

$$m_{12} = \frac{1}{2}\left(F_x^T - \bar{x}^T F_{xx}\right).$$

Notice that in the second line above we used the fact that F is a scalar and thus $F_x^T x$ is a scalar as well, but the scalar equals its own transpose.

LQ-ization VII

The remaining terms are swept into the constant m_{11}:

$$m_{11} = \bar{F} - F_x^T \bar{x} + \frac{1}{2} \bar{x}^T F_{xx} \bar{x}.$$

As it is easy to see, these are exactly the formulae we have had before.

LQ Problem after LQ-ization I

After this transformation, our problem can be written as

$$\max \sum_{t=0}^{\infty} \beta^t \left(1, x_t^T, u_t^T\right) M \begin{bmatrix} 1 \\ x_t \\ u_t \end{bmatrix},$$

$$\text{s.t.} \quad \begin{bmatrix} 1 \\ x_{t+1} \end{bmatrix} = A \begin{bmatrix} 1 \\ x_t \end{bmatrix} + Bu_t, \quad x_0 \text{ given.}$$

Now, the vector consisting of one and x plays the role of the state vector, and u is still the control vector.

LQ Problem after LQ-ization II

The only difference with the notation that we saw before is that we have a cross-product term in the objective function, which now could be written as

$$\beta^t \left(x_t^T R x_t + u_t^T Q u_t + 2 u_t^T W x_t \right).$$

The same guess $V(x) = x^T P x$ works, and repeating the steps perfomed before, we can get

$$u = -\left(Q + B^T \beta P B \right)^{-1} \left(W + B^T \beta P A \right) x,$$

$$P = R + A^T \beta P A - \left(A^T \beta P B + W^T \right) \left(Q + B^T \beta P B \right)^{-1} \left(W + B^T \beta P A \right).$$

Once again, P could be obtained in the limit of value function iterations.