Macro Topics: Introduction to Matlab

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Lecture notes (December 16)

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Topics Covered Today

Log-Linearization Techniques

- Basic RBC Model
- Writing the Model in a Stationary Form
- Solving the Model
- Log-Linearizing Simplified RBC Model
- Solving Log-Linearized Model

This lecture is partially based on McCandless book, Chapter 6.

Basic RBC Model I

RBC (Real Business Cycle) approach uses models that describe both growth and fluctuations.

- *Consumers:*

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(C_t, L_t),$$

$$\beta > 0,\ L_t \text{ is leisure.}$$

- $N_t + L_t = 1$: Time can be spent working or consuming the leisure.
- *Technology:*

$$Y_t = A_t F(K_t, N_t X_t),$$

$$X_{t+1} = X_t \gamma,\ \gamma > 1.$$

Here A_t is productivity shock, X_t is productivity growth term, or technical progress.

Basic RBC Model II

- X_t can be introduced as $Y_t = X_t^H F(K_t X_t^K, N_t X_t^N)$.
  - X_t^H is Hicks-neutral technical progress,
  - X_t^K — capital-augmenting,
  - X_t^N — labor-augmenting (Harrod-neutral).

  With Cobb-Douglas F, all 3 kinds are the same. For general CRS production function, only X_t^N generates constant "great ratios" along the balanced growth path.
- *One-Sector model:*

$$Y_t = C_t + I_t, \quad K_{t+1} = I_t + (1-\delta)K_t,$$

$$K_0,\ X_0(=1),\ A_0 \text{ given.}$$

Basic RBC Model III

- *Usual period utility function:*

$$U(C,L) = \frac{[C\nu(L)]^{1-\sigma} - 1}{1-\sigma}.$$

This generates (C^*, L^*) not depending on real wage as in many developed countries' data:

$$\max U(C,L)$$

$$\text{s.t.} \quad C = w(1-L),$$

$$FOC: \quad [C\nu(L)]^{-\sigma} w[\nu'(L)(1-L) - \nu(L)] = 0,$$

$$\Rightarrow \quad L^* \text{ does not depend on } w.$$

Writing the Model in a Stationary Form I

- Divide growing variables by X_t (all the variables are now in intensive form, except for L_t):

  $$c_t = \frac{C_t}{X_t}, \quad y_t = \frac{Y_t}{X_t}, \quad k_t = \frac{K_t}{X_t}, \quad i_t = \frac{I_t}{X_t}.$$

  The only non-standard changes:

  $$\sum \beta^t \frac{[C_t \nu(L_t)]^{1-\sigma} - 1}{1-\sigma} = \sum (\beta \gamma^{1-\sigma})^t U(c_t, L_t).$$

  (to obtain the result, we dropped/added unsignificant constants).

  $$K_{t+1} = I_t + (1-\delta) K_t \quad \Rightarrow \quad \frac{K_{t+1}}{X_t} = i_t + (1-\delta) k_t$$

  $$\Rightarrow \quad \frac{K_{t+1}}{X_{t+1}} \frac{X_{t+1}}{X_t} = k_{t+1} \gamma = i_t + (1-\delta) k_t$$

Writing the Model in a Stationary Form II

- The model then becomes

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} \underbrace{(\beta\gamma^{1-\sigma})^t}_{\text{will write just } \beta^t} U(c_t, L_t),$$

$$\text{s.t.} \quad \left.\begin{aligned} N_t &= 1 - L_t \\ y_t &= A_t F(k_t, N_t) \\ y_t &= c_t + i_t \\ \gamma k_{t+1} &= i_t + (1-\delta) k_t \end{aligned}\right\} \text{ or}$$

$$\gamma k_{t+1} = A_t F(k_t, 1 - L_t) + (1-\delta) k_t - c_t.$$

- Bellman equation is

$$V(k, A) = \max_{(c,N,k')} \{ U(c, 1-N) + \beta \mathbb{E} V(k', A') \}$$

$$\text{s.t.} \quad \gamma k' = AF(k, N) - c + (1-\delta) k.$$

- In reality, we have 2 choise (control) variables. The most convinient choise is N and k', with c given by the budget constraint.

Solving the Model I

$$V(k,A)=\max_{(N,k')}\{U(AF(k,N)+(1-\delta)k-\gamma k',1-N)+\beta\mathbb{E}V(k',A')\}$$

- FOC:

$$N:\quad U_1AF_2-U_2=0\quad\Rightarrow\quad U_1AF_2=U_2,$$

$$k':\quad U_1(-\gamma)+\beta\mathbb{E}V_1(k',A')=0.$$

- ET:

$$V_1(k,A)=U_1(AF_1(k,N)+1-\delta).$$

  A is "exogenous state", no V_2 in FOC $\Rightarrow$ ET on V_2 not derived.

- Plug ET into 2nd FOC to get EE:

$$-\gamma U_1(c,1-N)+\beta\mathbb{E}[U_1(c',1-N')(A'F_1(k',N')+1-\delta)]=0.$$

Solving the Model II

- Non-stochastic version ($A_t = \bar{A}$, $c_t = \bar{c}$, $N_t = \bar{N}$, $k_t = \bar{k}$) has a unique steady state:

  $$\beta U_1(\bar{A}F_1 + 1 - \delta) - \gamma U_1 = 0,$$

  $$\bar{A}F_1 = \frac{\gamma}{\beta} - 1 + \delta,$$

  meaning that marginal product of capital equals $r + \delta$, the rental rate of capital, and $r = \gamma/\beta - 1$.

Log-Linearizing Simplified RBC Model I

Why **log**-linearization?

1. Many macroeconomic variables behave as if they were growing with a stationary rate. After taking logs, they become approximately linear functions of time.
2. In logs, a regression coefficient in a regression of y on x gives directly the elasticity.
3. Many sensible functional forms lead to linear ot almost linear equations after taking logs.
4. Easy to interpret: log-deviation is approximately equal to the percentage deviation.
5. Very often, we are interested in the behavior of a model near some point. In a small region, all functions are approximately linear. Usually, we log-linearize around non-stochastic steady state.

Log-Linearizing Simplified RBC Model II

We will be working with a simplified model, with log utility of consumption, without labor supply decision but with productivity shocks, and no growth.

- We have the following equations:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} (1 - \delta + \alpha s_{t+1} k_{t+1}^{\alpha-1}) \right],$$

$$k_{t+1} = (1-\delta) k_t + s_t k_t^{\alpha} - c_t,$$

$$s_t = s_{t-1}^{\rho} v_t.$$

  Here v_t is an i.i.d. random variable with mean one.

- Our goal is to obtain a set of linear equations in log-deviations from the steady state, defined as

$$\tilde{c} = \ln \frac{c}{\bar{c}} = \ln \frac{c - \bar{c} + \bar{c}}{\bar{c}} = \ln \left(1 + \frac{c - \bar{c}}{\bar{c}} \right) \approx \frac{c - \bar{c}}{\bar{c}}.$$

Log-Linearizing Simplified RBC Model III

- Let's start working with the first equation:

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} (1 - \delta + \alpha s_{t+1} k_{t+1}^{\alpha-1}) \right],$$

$$\frac{\bar{c}}{c_t} = \mathbb{E}_t \left[\frac{\bar{c}}{c_{t+1}} \beta (1 - \delta + \alpha s_{t+1} k_{t+1}^{\alpha-1}) \right].$$

(bar over a variable denotes it's a steady state).

- Take logs of both sides (we will be linearizing anyway, this justifies taking log under E_t sign). Being more strict, we could just expand all the functions up to 1st order, take expectation, and only then take the logs.

$$\ln \frac{\bar{c}}{c_t} = -\ln \frac{c_t}{\bar{c}} = -\tilde{c}_t \ =$$

$$= \ \mathbb{E}_t \left[\ln \frac{\bar{c}}{c_{t+1}} + \ln\{\beta(1 - \delta + \alpha s_{t+1} k_{t+1}^{\alpha-1})\} \right] \ =$$

$$= \ -\mathbb{E}_t \tilde{c}_{t+1} + \mathbb{E}_t [\ln\{\beta(1 - \delta + \alpha s_{t+1} k_{t+1}^{\alpha-1})\}].$$

Log-Linearizing Simplified RBC Model IV

In the steady state,

$$\frac{1}{\bar{c}} = \frac{\beta}{\bar{c}}(1 - \delta + \alpha \bar{s}\bar{k}^{\alpha-1}) \quad \Rightarrow \quad 1 - \delta = \frac{1}{\beta} - \alpha \bar{s}\bar{k}^{\alpha-1},$$

plug this into the previous expression:

$$\mathbb{E}_t[\ln\{\beta(1-\delta+\alpha s_{t+1}k_{t+1}^{\alpha-1})\}] = \mathbb{E}_t[\ln(1-\alpha\beta\bar{s}\bar{k}^{\alpha-1}+\alpha\beta s_{t+1}k_{t+1}^{\alpha-1})] \approx$$

$$\approx \mathbb{E}_t[\alpha\beta(s_{t+1}k_{t+1}^{\alpha-1} - \bar{s}\bar{k}^{\alpha-1})]$$

and now do Taylor approximation of function inside:

$$s_{t+1}k_{t+1}^{\alpha-1} - \bar{s}\bar{k}^{\alpha-1} = 0 + \bar{k}^{\alpha-1}(s_{t+1} - \bar{s}) + (\alpha-1)\bar{s}\bar{k}^{\alpha-2}(k_{t+1} - \bar{k}) =$$

$$= \bar{s}\bar{k}^{\alpha-1}\frac{(s_{t+1} - \bar{s})}{\bar{s}} + (\alpha - 1)\bar{s}\bar{k}^{\alpha-1}\frac{(k_{t+1} - \bar{k})}{\bar{k}}.$$

Combining all this, we get

$$-\tilde{c}_t = \mathbb{E}_t[-\tilde{c}_{t+1} + \alpha\beta\bar{s}\bar{k}^{\alpha-1}\tilde{s}_{t+1} + \alpha\beta(\alpha - 1)\bar{s}\bar{k}^{\alpha-1}\tilde{k}_{t+1}].$$

Log-Linearizing Simplified RBC Model V

- Budget constraint:

$$k_{t+1} = (1-\delta)k_t + s_t k_t^{\alpha} - c_t,$$

in the steady state,

$$\bar{k} = (1-\delta)\bar{k} + \bar{s}\bar{k}^{\alpha} - \bar{c},$$

therefore

$$\frac{k_{t+1} - \bar{k}}{\bar{k}} = (1-\delta)\frac{k_t - \bar{k}}{\bar{k}} + \frac{s_t k_t^{\alpha} - \bar{s}\bar{k}^{\alpha}}{\bar{k}} - \frac{(c_t - \bar{c})}{\bar{c}}\frac{\bar{c}}{\bar{k}},$$

$$\frac{s_t k_t^{\alpha} - \bar{s}\bar{k}^{\alpha}}{\bar{k}} = \frac{\bar{k}^{\alpha}(s_t - \bar{s})}{\bar{s}}\frac{\bar{s}}{\bar{k}} + \frac{\alpha\bar{s}\bar{k}^{\alpha-1}(k_t - \bar{k})}{\bar{k}} =$$

$$= \bar{s}\bar{k}^{\alpha-1}\tilde{s}_t + \alpha\bar{s}\bar{k}^{\alpha-1}\tilde{k}_t,$$

$$\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \alpha\bar{s}\bar{k}^{\alpha-1}\tilde{k}_t + \bar{s}\bar{k}^{\alpha-1}\tilde{s}_t - \frac{\bar{c}}{\bar{k}}\tilde{c}_t.$$

Log-Linearizing Simplified RBC Model VI

- Finally, the law of motion for the exogenous shock is trivial:

$$\ln s_t = \rho \ln s_{t-1} + \ln v_t \quad \Rightarrow \quad \tilde{s}_{t+1} = \rho \tilde{s}_t + \tilde{v}_{t+1}.$$

- Thus, the 3 basic equations

$$\frac{1}{c_t} = \beta \mathbb{E}_t \left[\frac{1}{c_{t+1}} (1 - \delta + \alpha s_{t+1} k_{t+1}^{\alpha-1}) \right],$$

$$k_{t+1} = (1-\delta) k_t + s_t k_t^{\alpha} - c_t,$$

$$s_t = s_{t-1}^{\rho} v_t$$

become after log-linearization,

$$-\tilde{c}_t = \mathbb{E}_t[-\tilde{c}_{t+1} + a_1 \tilde{k}_{t+1} + a_2 \tilde{s}_{t+1}],$$

$$\tilde{k}_{t+1} = b_1 \tilde{k}_t + b_2 \tilde{s}_t + b_3 \tilde{c}_t,$$

$$\tilde{s}_{t+1} = \rho \tilde{s}_t + \tilde{v}_{t+1}.$$

Log-Linearizing Simplified RBC Model VII

- We can write this as

$$\begin{pmatrix} -1 & 0 & 0 \\ b_3 & b_1 & b_2 \\ 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{pmatrix} = \begin{pmatrix} -1 & a_1 & a_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{pmatrix} + $$

$$+ \begin{pmatrix} 0 & -1 & a_1 & a_2 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \tilde{v}_{t+1} \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{pmatrix},$$

  where $w^x_{t+1} = \mathbb{E}_t[x_{t+1}] - x_{t+1}$ for $x = c, k, s$ (expectational errors).

Log-Linearizing Simplified RBC Model VIII

Pre-multiplying this system by $\begin{pmatrix} -1 & 0 & 0 \\ b_3 & b_1 & b_2 \\ 0 & 0 & \rho \end{pmatrix}^{-1}$, we get

$$\tilde{y}_t = \begin{pmatrix} \tilde{c}_t \\ \tilde{k}_t \\ \tilde{s}_t \end{pmatrix} = A \begin{pmatrix} \tilde{c}_{t+1} \\ \tilde{k}_{t+1} \\ \tilde{s}_{t+1} \end{pmatrix} + B \begin{pmatrix} \tilde{v}_{t+1} \\ w^c_{t+1} \\ w^k_{t+1} \\ w^s_{t+1} \end{pmatrix} = A\tilde{y}_{t+1} + B \begin{pmatrix} \tilde{v}_{t+1} \\ w_{t+1} \end{pmatrix}.$$

Solving Log-Linearized Model I

- How to solve these equations?
- Diagonalize A:

$$A = Q\Lambda Q^{-1},$$

where Λ diagonal (in economics, eigenvalues are usually distinct $\Rightarrow$ Q consists of eigenvectors, and we don't need Jordan form. However, the same argument will work with the Jordan form also).

- Pre-multiply by Q^{-1} to get

$$\tilde{z}_t = Q^{-1}\tilde{y}_t = \Lambda Q^{-1}\tilde{y}_{t+1} + Q^{-1}B\begin{pmatrix}\tilde{v}_{t+1}\\ w_{t+1}\end{pmatrix} = \Lambda\tilde{z}_{t+1} + Q^{-1}B\begin{pmatrix}\tilde{v}_{t+1}\\ w_{t+1}\end{pmatrix}$$

$$\mathbb{E}_t Q^{-1}B\begin{pmatrix}\tilde{v}_{t+1}\\ w_{t+1}\end{pmatrix} = 0 \quad \Rightarrow \quad \tilde{z}_t^i = \lambda_i \mathbb{E}_t[\tilde{z}_{t+1}^i],$$

a set of disjoint equations.

Solving Log-Linearized Model II

- Why is $\mathbb{E}_t Q^{-1} B \begin{pmatrix} \tilde{v}_{t+1} \\ w_{t+1} \end{pmatrix}$ equal to zero? Because expectation of $\tilde{v}_{t+1}$ is zero, and expectations are correct on average by the Rational Expectations (RE) Hypothesis. In fact, RE says that subjective probability distributions are equal to the true distributions, and therefore all subjective moments are equal to their theoretical counterparts. Then,
$$\mathbb{E}_t[\mathbb{E}_t[x_{t+1}] - x_{t+1}] = 0.$$
- Iterating the equation $\tilde{z}_t^i = \lambda_i \mathbb{E}_t[\tilde{z}_{t+1}^i]$ forward T times and using the law of iterated expectations ($\mathbb{E}_t \mathbb{E}_{t+1} \mathbb{E}_{t+2} ... = \mathbb{E}_t$), we get
$$\tilde{z}_t^i = \lambda_i^T \mathbb{E}_t[\tilde{z}_{t+T}^i].$$
If for a particular i we have $|\lambda_i| < 1$, then $\lim_{T \to \infty} \lambda^T = 0 \Rightarrow \tilde{z}_t^i = 0$ (we are working in a small neighborhood of the deterministic steady state $\Rightarrow \tilde{z}_{t+T}^i$ should be bounded for all T).

Solving Log-Linearized Model III

- In general, matrix A might have n_s roots within unit circle, which generates n_s restrictions of the form $\tilde{z}_t^i = 0$.
- In our optimal growth example, there is one stable root, and one non-predetermined variable $\tilde{c}_t$ which could be adjusted to solve the equation $\tilde{z}_t^i = 0$.
- What does it mean to solve $\tilde{z}_t^i = 0$? Remember that $\tilde{z}_t = Q^{-1}\tilde{y}_t$, and thus $\tilde{z}_t^i$ equals ith row of Q^{-1} times $\tilde{y}_t$: $(Q^{-1}\tilde{y}_t)^i = 0$.
- This gives a linear relationship among $\tilde{c}_t$, $\tilde{k}_t$, and $\tilde{s}_t$,

$$\tilde{c}_t = q^{i1}\tilde{k}_t + q^{i2}\tilde{s}_t.$$

Solving Log-Linearized Model IV

- Plug this $\tilde{c}_t$ into the log-linearized budget constraint, and get the VAR representation of the model:

$$\begin{pmatrix} \tilde{k}_t \\ \tilde{s}_t \end{pmatrix} = \bar{A} \begin{pmatrix} \tilde{k}_{t-1} \\ \tilde{s}_{t-1} \end{pmatrix} + \bar{b}\tilde{v}_t.$$

Now given $\tilde{k}_0$, $\tilde{s}_0$ and a sequence of innovations $\{\tilde{v}_t\}$, one could simulate the model, generating a triple of time series $\{\tilde{k}_t, \tilde{s}_t, \tilde{c}_t\}_{t=0}^{\infty}$.