Macro Topics: Introduction to Matlab

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Lecture notes (November 22)

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Topics Covered Today

Solution Methods: Guess and Verify, Analytical Value Function Iterations, Policy Iterations

- Log Utility, Cobb-Douglas Production function, Full Depreciation Model
- Simplified Growth Model: Guess & Verify
- Simplified Growth Model: Analytical Value Function Iterations
- Simplified Growth Model: Analytical Policy Function Iterations (Howard Improvement Algorithm)
- Search/Stop Problem

This lecture is based on Ljungquist & Sargent book, Chapters 3 and 6.3.

Log Utility, Cobb-Douglas Production function, Full Depreciation Model

Consider again our Robinson Crusoe problem. For specific functional forms, the problem could be solved analytically using several methods. The functional forms are: utility logarithmic in consumption, Cobb-Douglas production function, and full depreciation of capital in a period:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c_t,$$

$$\text{subject to} \quad k_{t+1} = Ak_t^{\alpha} - c_t, \;\; 0 < \alpha < 1,$$

$$c_t > 0, \;\; k_t > 0, \;\; k_0 \text{ given.}$$

The Bellman equation is

$$V(k) = \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta V\left(\tilde{k}\right) \right\}.$$

Simplified Growth Model: Guess & Verify I

Make a (very lucky) guess that $V(k) = \Psi + \Phi \ln k$. Need to find Ψ, Φ.

FOC:

$$\left[\ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta\left(\Psi + \Phi \ln \tilde{k}\right)\right]' = 0,$$

$$-\frac{1}{Ak^{\alpha} - \tilde{k}} + \frac{\beta\Phi}{\tilde{k}} = 0,$$

$$\beta\Phi Ak^{\alpha} - \beta\Phi\tilde{k} = \tilde{k},$$

$$\tilde{k} = \frac{\beta\Phi}{1 + \beta\Phi} Ak^{\alpha}.$$

Simplified Growth Model: Guess & Verify II

ET:

$$V'(k) = \frac{\partial \ln\left(Ak^{\alpha} - \tilde{k}\right)}{\partial k} \quad \Rightarrow \quad \frac{\Phi}{k} = \frac{1}{Ak^{\alpha} - \tilde{k}} \alpha A k^{\alpha-1},$$

$$\Phi A k^{\alpha} - \Phi \tilde{k} = \alpha A k^{\alpha} \quad \xrightarrow{\text{FOC}} \quad (\Phi - \alpha) A k^{\alpha} = \Phi \frac{\beta\Phi}{1+\beta\Phi} A k^{\alpha},$$

$$\beta\Phi^2 = \Phi + \beta\Phi^2 - \alpha - \beta\alpha\Phi \quad \Rightarrow \quad \Phi = \frac{\alpha}{1-\beta\alpha},$$

$$\xrightarrow{\text{back to FOC}} \quad \tilde{k} = \beta\alpha A k^{\alpha}.$$

We used method of undetermined coefficients: we collected terms with k^{α} and equated their coefficients on LHS and RHS.

Simplified Growth Model: Guess & Verify III

Now, plug everything back into Bellman equation:

$$V = \Psi + \frac{\alpha}{1-\beta\alpha}\ln k \ =$$

$$= \ \ln\left(Ak^{\alpha} - \beta\alpha Ak^{\alpha}\right) + \beta\left(\Psi + \frac{\alpha}{1-\beta\alpha}\ln\left(\beta\alpha Ak^{\alpha}\right)\right) \ =$$

$$= \ \ln A(1-\beta\alpha) + \alpha\ln k + \beta\left(\Psi + \frac{\alpha}{1-\beta\alpha}\ln(\beta\alpha A)\right) + \beta\frac{\alpha}{1-\beta\alpha}\alpha\ln k,$$

and equate constants (note that terms with $\ln k$ on LHS and RHS cancel out):

$$\Psi = \beta\Psi + \ln A(1-\beta\alpha) + \frac{\beta\alpha}{1-\beta\alpha}\ln(A\beta\alpha),$$

$$\Psi = \frac{1}{1-\beta}\left[\ln A(1-\beta\alpha) + \frac{\beta\alpha}{1-\beta\alpha}\ln(A\beta\alpha)\right].$$

Simplified Growth Model: Guess & Verify IV

Both Ψ and Φ are now determined as functions of model's parameters, which means our guess was correct. With incorrect guess of the functional form of V, we wouldn't be able to derive the coefficients.

Simplified Growth Model: Analytical Value Function Iterations I

The basic idea is to use the contraction property of the Bellman operator T and find solution by consequtive iterations.

(1) Start with $V_0(k) = 0$. The problem becomes

$$V_1(k) = \max_{\tilde{k}} \left\{ \ln \left(Ak^{\alpha} - \tilde{k} \right) + \beta \cdot 0 \right\},$$

$$\tilde{k} = 0 \text{ as } \ln \text{ is increasing function,}$$

$$V_1(k) = \ln A + \alpha \ln k.$$

Simplified Growth Model: Analytical Value Function Iterations II

(2) At the second step, we have

$$V_2(k) = \max_{\tilde{k}} \left\{ \ln \left(Ak^\alpha - \tilde{k} \right) + \beta V_1 \left(\tilde{k} \right) \right\},$$

$$V_2(k) = \max_{\tilde{k}} \left\{ \ln \left(Ak^\alpha - \tilde{k} \right) + \beta \left(\ln A + \alpha \ln \tilde{k} \right) \right\},$$

$$-\frac{1}{Ak^\alpha - \tilde{k}} + \frac{\alpha\beta}{\tilde{k}} = 0 \quad \Rightarrow \quad \tilde{k} = \alpha\beta \left(Ak^\alpha - \tilde{k} \right) \quad \Rightarrow$$

$$\Rightarrow \quad \tilde{k} = \frac{\alpha\beta Ak^\alpha}{1+\alpha\beta},$$

$$V_2(k) = \ln \frac{Ak^\alpha}{1+\alpha\beta} + \beta \ln A + \alpha\beta \ln \frac{\alpha\beta Ak^\alpha}{1+\alpha\beta} =$$

$$= \ln \frac{A}{1+\alpha\beta} + \beta \ln A + \alpha\beta \ln \frac{\alpha\beta A}{1+\alpha\beta} + \alpha(1+\alpha\beta) \ln k.$$

Simplified Growth Model: Analytical Value Function Iterations III

(3) The third step then gives for $V_3(k)$:

$$\max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta V_2\left(\tilde{k}\right) \right\} =$$

$$= \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \alpha\beta(1+\alpha\beta)\ln\tilde{k} + const \right\},$$

$$0 = -\frac{1}{Ak^{\alpha} - \tilde{k}} + \frac{\alpha\beta(1+\alpha\beta)}{\tilde{k}} \quad \Rightarrow \quad \tilde{k} = \frac{(\alpha\beta + \alpha^2\beta^2)Ak^{\alpha}}{1+\alpha\beta+\alpha^2\beta^2},$$

$$V_3(k) = \ln\frac{A}{1+\alpha\beta+\alpha^2\beta^2} + \alpha\ln k + \beta\ln\frac{A}{1+\alpha\beta} + \alpha\beta^2\ln\frac{\alpha\beta A}{1+\alpha\beta} +$$

$$+ \alpha\beta(1+\alpha\beta)\ln\frac{\left(\alpha\beta+\alpha^2\beta^2\right)A}{1+\alpha\beta+\alpha^2\beta^2} + \alpha\beta(1+\alpha\beta)\ln k.$$

Simplified Growth Model: Analytical Value Function Iterations IV

(4) This procedure could be repeated further, but it is already becoming obvious that the optimal policy looks like

$$\tilde{k} = \frac{\alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3 + \dots}{1 + \alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3 + \dots} A k^{\alpha},$$

and going to the limit we get familiar-looking policy function

$$\tilde{k} = \alpha\beta A k^{\alpha}.$$

Simplified Growth Model: Analytical Value Function Iterations V

(5) It is also possible to notice a pattern in $V_k(k)$ (a constant and $\ln k$) and go to the limit, but the easier way is just to plug in the limiting policy function and use the method of undetermined coefficients again:

$$\Psi + \Phi \ln k = \ln[1 - \beta\alpha]Ak^{\alpha} + \beta(\Psi + \Phi \ln(\alpha\beta A k^{\alpha})),$$

$$const: \quad \Psi = \ln[1 - \beta\alpha]A + \beta\Psi + \beta\Phi \ln(\alpha\beta A),$$

$$\ln k: \quad \Phi = \alpha + \beta\Phi\alpha.$$

(6) First solving the equation for Φ gives familiar $\Phi = \alpha/(1-\beta\alpha)$, and then Ψ is the same as in the Guess & Verify case.

Simplified Growth Model: Analytical Policy Function Iterations (Howard Improvement Algorithm) I

Assume the candidate policy function $\tilde{k}_1 = h_0 A k^{\alpha}$.

(1) Plugging the policy function into the Bellman equation (policy evaluation step), we get

$$V_1(k) = \ln(Ak^{\alpha} - h_0 A k^{\alpha}) + \beta V_1(h_0 A k^{\alpha}),$$

$$V_1(k) = \ln(1 - h_0)A + \alpha \ln k + \beta V_1(h_0 A k^{\alpha}).$$

Simplified Growth Model: Analytical Policy Function Iterations (Howard Improvement Algorithm) II

(2) It now becomes immediately obvious that the value function has the form $V_1(k) = \Psi + \Phi \ln k$. Plugging this into the above equation and using the method of undetermined coefficients, we get

$$\Psi + \Phi \ln k = \ln(1 - h_0)A + \alpha \ln k + \beta\Psi + \beta\Phi \ln(h_0 A) + \alpha\beta\Phi \ln k,$$

which again gives $\Phi = {}^{\alpha}/_{(1-\beta\alpha)}$. Then,

$$\Psi = \ln(1 - h_0)A + \beta\Psi + \beta\frac{\alpha}{1 - \alpha\beta}\ln(h_0 A),$$

$$\Psi = \frac{1}{1 - \beta}\left(\ln(1 - h_0)A + \frac{\alpha\beta}{1 - \alpha\beta}\ln(h_0 A)\right).$$

Simplified Growth Model: Analytical Policy Function Iterations (Howard Improvement Algorithm) III

(3) In the policy improvement step, we have to perform maximization:

$$V(k) = \max_{\tilde{k}} \left\{ \ln \left(Ak^{\alpha} - \tilde{k} \right) + \beta \left[\Psi + \frac{\alpha}{1 - \alpha\beta} \ln \tilde{k} \right] \right\},$$

$$\frac{1}{Ak^{\alpha} - \tilde{k}} = \frac{\alpha\beta}{1 - \alpha\beta} \frac{1}{\tilde{k}}, \quad \tilde{k}_2 = \alpha\beta A k^{\alpha}.$$

(4) We could, in principle, perform further policy evaluation and improvement step, but $\tilde{k}_3$ will be exactly the same as $\tilde{k}_2$. The policy improvement algorithm, therefore, converged in just one step.

Simplified Growth Model: Analytical Policy Function Iterations (Howard Improvement Algorithm) IV

(5) This is a general result: policy improvement algorithm, as a rule, converges much faster than value function iterations, even if the initial guess regarding the functional form is not correct (here we guessed correctly that $\tilde{k} \sim Ak^{\alpha}$, or is a fixed share of output). Still, notice that in the value function iterations case, even with a correct guess we needed infinite number of iterations to converge.

Search/Stop Problem I

- At every period t consumer draws job offer $x \sim U[0,1]$.
- If she accepts, termination payoff is x (can think of an offer as a NPV of the sequence of wage payments).
- Otherwise, continue and draw another offer next period. (So, there are only two actions available to the agent, and maximization is reduced to selecting the best one).
- Offers are i.i.d. across periods, that is, offer today and offer tomorrow are not correlated.
- Discount factor is β.
- Bellman's equation:

$$V(x) = \max \left\{ \underset{\substack{\uparrow \\ stop}}{x}, \beta \mathbb{E}[\underset{\substack{\uparrow \\ continue}}{V(\tilde{x})}|x] \right\}.$$

Search/Stop Problem II

- It's easy to see that for every state x, we are choosing between an unknown number $\beta\mathbb{E}[V(\tilde{x})|x]$ and a linearly increasing function x. Therefore, an easy guess is that optimal policy is given as

$$CONTINUE \quad if\ x \leq x^*,$$

$$STOP \quad if\ x > x^*,$$

  for some, yet unknown, x^*. x^* is NPV of the *reservation wage*.
- At x^*, value from both actions should coincide. Also notice that x^* is both continuation payoff and cutoff threshold.

$$V(x) = \begin{cases} x^*, & if\ x \leq x^* \\ x, & if\ x > x^* \end{cases}$$

- Note that if consumer rejects the offer, her value does not depend on its value and is constant. This is because the offers are i.i.d., and future offer(s) do not depend on the one that was rejected.

Search/Stop Problem III

- Now, let us start the iterations. Take the easiest initial value function, zero, $V_0 \equiv 0$:

$$(TV_0)(x) = \max\left\{x, \beta\mathbb{E}[\underbrace{V_0(\tilde{x})}_{=0}]\right\} = x,$$

$$V^1(x) = \begin{cases} x^{[1]}, & \textit{if } x \leq x^{[1]} \\ x, & \textit{if } x > x^{[1]} \end{cases}, \quad x^{[1]} = 0.$$

After the first iteration our optimal policy is to STOP (accept offer) always.

Search/Stop Problem IV

- At the next iteration,

$$(T^2V_0)(x) = \max\{x, \beta\mathbb{E}[TV_0(\tilde{x})]\} = \max\{x, \beta\mathbb{E}[\tilde{x}]\} \quad \Rightarrow$$

$$(T^2V_0)(x) = \begin{cases} x^{[2]}, & \textit{if } x \leq x^{[2]} \\ x, & \textit{if } x > x^{[2]} \end{cases}, \quad x^{[2]} = \frac{1}{2}\beta,$$

because expected value of a variable that is uniformly distributed on $(0,1)$ is just one half.

After the second iteration, the optimal policy is to STOP if offer is above $x^{[2]} = {}^1\!/_2\beta$ and CONTINUE otherwise.

Search/Stop Problem V

- At a generic iteration, we have

$$(T^n V_0)(x) = \max\{x, \beta \mathbb{E}[(T^{n-1} V_0)(\tilde{x})]\},$$

$$x^{[n]} = \beta \mathbb{E}[(T^{n-1} V_0)(\tilde{x})],$$

$$(T^n V_0)(x) = \begin{cases} x^{[n]}, & \text{if } x \leq x^{[n]} \\ x, & \text{if } x > x^{[n]} \end{cases},$$

$$(T^{n-1} V_0)(x) = \begin{cases} x^{[n-1]}, & \text{if } x \leq x^{[n-1]} \\ x, & \text{if } x > x^{[n-1]} \end{cases}.$$

Note that the value function from the initial guess is of the same form as value functions after any number of iterations: first, it is a constant, and then it linearly increases with x. Convergence of $T^n V_0$ to V, the solution of the Bellman equation, is then equivalent to the convergence of cutoff thresholds $x^{[n]}$ to some limit x^*.

Search/Stop Problem VI

- Now we know the general form of the value function, and can derive the relationship between $x^{[n]}$ and $x^{[n-1]}$:

$$x^{[n]} = \beta \mathbb{E}[(T^{n-1}V_0)(\tilde{x})] =$$

$$= \beta \left[\int_0^{x^{[n-1]}} x^{[n-1]} f(x) dx + \int_{x^{[n-1]}}^1 x f(x) dx \right] = \frac{1}{2}\beta \left[1 + \left(x^{[n-1]} \right)^2 \right].$$

We have derived a difference equation, $x^{[n]} = f(x^{[n-1]})$. To find x^*, set $x^{[n]} = x^{[n-1]} = x^*$, to get

$$x^* = \frac{1}{2}\beta \left[1 + (x^*)^2 \right] \quad \Rightarrow \quad (x^*)^2 - \frac{2}{\beta} x^* + 1 = 0.$$

Positive root of this quadratic equation is given by

$$\frac{1}{\beta} - \sqrt{\frac{1}{\beta^2} - 1} = \frac{1}{\beta} \left(1 - \sqrt{1 - \beta^2} \right).$$

Search/Stop Problem VII

- Note that the solution was actually a combination of guess and verify method and analytical iterations, because we had to guess how the optimal policy looked like (we guessed that it looks like a reservation wage policy).