Macro Topics: Introduction to Matlab
Fall semester 2016

Lecture notes (November 18)

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Topics Covered Today

Infinite Horizon Dynamic Programming: Ljungquist & Sargent Notation

- Deterministic Case
- Properties of the Solution of the Bellman Equation: Deterministic Case
- Stochastic Case
- Properties of the Solution of the Bellman Equation: Stochastic Case
- Consumption and Savings: Basic Growth Model
- Consumption and Savings: Bellman Equation Deriviation
- Consumption and Savings: Solution

This lecture is based on Ljungquist & Sargent book, Chapter 3, and McCandless book, Chapter 3.

Infinite Horizon DP. Ljungquist & Sargent Notation. Deterministic Case I

The dynamic programming problem we will be dealing with in economics (as written in Ljungquist & Sargent) is

$$V(x_0) = \max_{\{u_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t r(x_t, u_t),$$

$$\text{s.t.} \quad x_{t+1} = g(x_t, u_t), \quad x_0 \text{ given.}$$

Bellman equation:

$$V(x) = \max_u \{r(x, u) + \beta V(g(x, u))\}$$

Infinite Horizon DP. Ljungquist & Sargent Notation. Deterministic Case II

We are looking for the time-invariant policy function

$$u = h(x)$$

and the value function $V(x)$ which solves the Bellman equation.

Standard assumptions: r is continuous, concave, and bounded, and the set

$$\{\tilde{x}, x, u : \tilde{x} \leq g(x, u)\}$$

is convex and compact.

Note: tilda over a variable usually denotes next period's variable, or $t+1$. Variables without tilda are today's variables, or time t.

Infinite Horizon DP. L&S Notation. Properties of the Solution of the BE. Deterministic Case I

(1) $\exists!$ strictly concave solution $V(x)$.

(2) $V(x)$ could be derived by value function iterations

$$V_{j+1}(x) = \max_u \{r(x,u) + \beta V_j(\tilde{x})\},$$

starting from any bounded V_0.

(3) $\exists!$ time-invariant policy function $u = h(x)$, where h maximizes the RHS of BE. For well-behaved functions, this means that the optimal policy function $h(x)$ satisfies the first-order condition, or FOC:

$$\frac{\partial r[x, h(x)]}{\partial u} + \beta V'\{g[x, h(x)]\} \frac{\partial g[x, h(x)]}{\partial u} = 0,$$

$$r_2 + \beta V'(\tilde{x}) g_2 = 0.$$

In case x and u are vectors, g_2 is a matrix while V' is a row vector (V itself is a scalar).

Infinite Horizon DP. L&S Notation. Properties of the Solution of the BE. Deterministic Case II

(4) In the interior, the value function V is differentiable, and $V'(x)$ is given by (the Envelope Theorem, or ET condition)

$$V'(x) = \frac{\partial r[x, h(x)]}{\partial x} + \beta V'\{g[x, h(x)]\} \frac{\partial g[x, h(x)]}{\partial x} =$$

$$= r_1 + \beta V'(\tilde{x}) g_1.$$

Heuristic derivation of ET:

$$V'(x) = r_1 + r_2 h' + \beta V'(\tilde{x})(g_1 + g_2 h') =$$

$$= r_1 + \beta V'(\tilde{x}) g_1 + \underbrace{(r_2 + \beta V'(\tilde{x}) g_2)}_{\equiv 0 \text{ by FOC}} h' = r_1 + \beta V'(\tilde{x}) g_1.$$

Informally, h is chosen optimally given x, and so a small variation in x shouldn't influence the value ($h' = 0$ at the point of max), therefore there is no effect on V from the fact that $u = h(x)$ changes when x shifts.

Infinite Horizon DP. L&S Notation. Properties of the Solution of the BE. Deterministic Case III

(5) Very often it is possible to select the state and the control variables so that ${}^{\partial g}/_{\partial x} \equiv 0$. In particular, this is true for a standard case where $\tilde{x} = g(x, u) = u$, or next period state is the control variable. In this case $g_2 = 1$,

$$FOC: \quad r_2 + \beta V'(\tilde{x}) = 0,$$

$$ET: \quad V' = r_1.$$

(6) Finally, we iterate ET condition forward and plug it into the FOC to get the Euler Equation:

$$EE: \quad r_2 + \beta \tilde{r}_1 = 0,$$

$$r_2(x, u) + \beta r_1(\tilde{x}, \tilde{u}) = 0.$$

Infinite Horizon DP. Ljungquist & Sargent Notation. Stochastic Case

The problem is

$$V(x_0) = \max_{\{u_t\}_{t=0}^{\infty}} \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t r(x_t, u_t)\right],$$

$$\text{s.t.} \quad x_{t+1} = g(x_t, u_t, \varepsilon_{t+1}), \quad x_0 \text{ given.}$$

At time t, x_t is known, but not x_{t+j}, $j > 0$.

Bellman equation:

$$V(x) = \max_u \{r(x, u) + \beta \mathbb{E}[V(g(x, u, \varepsilon))|x]\}$$

Infinite Horizon DP. L&S Notation. Properties of the Solution of the BE. Stochastic Case I

(1) $\exists!$ strictly concave solution $V(x)$.

(2) $V(x)$ could be derived by value function iterations

$$V_{j+1}(x) = \max_u \{r(x,u) + \beta \mathbb{E}[V_j(\tilde{x})|x]\},$$

starting from any bounded V_0.

(3) $\exists!$ time-invariant policy function $u = h(x)$, where h maximizes the RHS of BE. For well-behaved functions, this means that the optimal policy function $h(x)$ satisfies the first-order condition, or FOC:

$$\frac{\partial r[x,h(x)]}{\partial u} + \beta \mathbb{E}\left[V'\{g[x,h(x)]\}\frac{\partial g[x,h(x)]}{\partial u}|x\right] = 0,$$

$$r_2 + \beta \mathbb{E}[V'(\tilde{x})g_2|x] = 0.$$

Infinite Horizon DP. L&S Notation. Properties of the Solution of the BE. Stochastic Case II

(4) In the interior, the value function V is differentiable, and $V'(x)$ is given by

$$V'(x) = \frac{\partial r[x, h(x)]}{\partial x} + \beta \mathbb{E}\left[V'\{g[x, h(x)]\} \frac{\partial g[x, h(x)]}{\partial x} | x \right] =$$

$$= \ r_1 + \beta \mathbb{E}[V'(\tilde{x}) g_1 | x].$$

(5) In case $g_1 = 0$, $g_2 = 1$, we still get $V' = r_1$.
EE becomes

$$r_2(x, u) + \beta \mathbb{E}[r_1(\tilde{x}, \tilde{u}) | x] = 0.$$

Consumption and Savings: Basic Growth Model I

Consider an economy with one agent: Robinson Crusoe. He wants to maximize a lifetime utility

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

where utility function u is continuous, increasing, concave, and has as many derivatives as needed.

Justification for infinite horizon:

(a) with realistic β (0.96 per year), utilities beyond typical lifespan are heavily discounted ($\beta^{50} \approx 0.13$), making infinite horizon solution a good approximation and

(b) if agent cares about utility of descendants (bequest motive), time horizon becomes effectively infinite.

Therefore, could consider utility as utility of a dynasty.

Consumption and Savings: Basic Growth Model II

The resource contraints are

$$k_{t+1} = (1-\delta)k_t + i_t,$$

$$y_t = f(k_t) = c_t + i_t.$$

Capital tomorrow k_{t+1} equals today's capital net of depreciation $(1-\delta)k_t$, plus investment i_t. This is a one good economy, and the single good is used both for consumption (c_t) and investment.

Given that this is a closed economy model, savings equal investment.

There is no uncertainty (deterministic problem).

Consumption and Savings: Basic Growth Model III

Formally, our problem now is: choose an infinite sequence of controls $\{c_t\}_{t=0}^{\infty}$, to maximize

$$\sum_{t=0}^{\infty} \beta^t u(c_t),$$

$$\text{subject to} \quad k_{t+1} = (1-\delta)k_t + f(k_t) - c_t, \tag{1}$$

$$c_t > 0, \quad k_t > 0, \quad k_0 \text{ given.}$$

Here, one can argue that k_t is state, c_t is control, and (1) is the transition law.

Consumption and Savings: Basic Growth Model IV

The Bellman equation could be written as

$$V(k) = \max_{c}\{u(c) + \beta V((1-\delta)k + f(k) - c)\},$$

but as usual it is easier to select next period state as control and then to express c from the resource constraint:

$$V(k) = \max_{\tilde{k}}\{u((1-\delta)k + f(k) - \tilde{k}) + \beta V(\tilde{k})\}.$$

Consumption and Savings: Bellman Equation Derivation I

Define value function as the maximized value of utility, which depends on the current value of capital (i.e. our state variable):

$$V_0(k_0) = \max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t).$$

Then take the current consumption out of the infinite sum to get:

$$V_0(k_0) = \max_{\{c_t\}_{t=0}^{\infty}} \left\{ u(c_0) + \sum_{t=1}^{\infty} \beta^t u(c_t) \right\}.$$

We can pass the maximization operator inside of the brackets:

$$V_0(k_0) = \max_{c_0} \left\{ u(c_0) + \max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^t u(c_t) \right\}.$$

Now note that the second term can be rewritten as

$$\beta \max_{\{c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t).$$

Consumption and Savings: Bellman Equation Derivation II

But this is actually the value function starting from period 1 (instead of period zero) multiplied by β. To see that, note that the value function starting from period 1 is defined as

$$V_1(k_1) = \max_{\{c_t\}_{t=1}^{\infty}} [u(c_1) + \beta u(c_2) + \beta^2 u(c_3) + ...].$$

Hence overall we get

$$V(k_0) = \max_{c_0} \{u(c_0) + \beta V(k_1)\}.$$

This is the Bellman equation, which states the problem as recursive problem, where we make choice over current period and all future periods.

Consumption and Savings: Solution I

The FOC and ET now are

$$FOC: \quad -u'(c) + \beta V'(\tilde{k}) = 0,$$

$$ET: \quad V'(k) = u'(c)(1 - \delta + f'(k)).$$

The Euler equation is

$$u'(c) = \beta u'(\tilde{c})(1 - \delta + f'(\tilde{k})).$$

Consumption and Savings: Solution II

Interpretation of the Euler Equation: if we are at the optimal trajectory, shifting Δc units of consumption into investment today (increasing capital by $\Delta k = \Delta c$) and consuming extra capital plus extra output tomorrow should leave utility approximately unchanged.

Utility drop today:

$$\Delta u \approx u'(c)\Delta c.$$

Tomorrow, we could consume non-depreciated extra capital $(1-\delta)\Delta k$ plus extra output $\approx f'(\tilde{k})\Delta k$. After discounting, this brings us extra utility

$$\Delta \tilde{u} \approx \beta u'(\tilde{c})(1-\delta+f'(\tilde{k}))\Delta k.$$

Equating Δu and $\Delta \tilde{u}$ gives the Euler Equation.

Consumption and Savings: Solution III

We could substitute consumption into the Euler Equation to get:

$$u'((1-\delta)k + f(k) - \tilde{k}) = \beta u'((1-\delta)\tilde{k} + f(\tilde{k}) - \tilde{\tilde{k}})(1-\delta + f'(\tilde{k})).$$

The result is a second order nonlinear difference equation in k (double tilda denotes a variable two periods into the future).

In the steady state (denoted by bar over a variable), we have

$$\frac{u'(\bar{c})}{u'(\bar{c})}\frac{1}{\beta} = \frac{1}{\beta} = 1 - \delta + f'(\bar{k}).$$

In the steady state, the marginal product of capital $f'(\bar{k})$ (rental rate when we are talking about decentralized version) equals net interest rate $1/\beta - 1$ plus depreciation rate δ.

Consumption and Savings: Solution without Substitution I

We can add the constraint at the end of the standard Bellman Equation with Lagrange multiplier as we would do in situation when we would not set up the problem in terms of dynamic programming:

$$V(k) = \max_{c,\tilde{k}} \{u(c) + \beta V(\tilde{k}) - \lambda_t(c + \tilde{k} - f(k) - (1-\delta)k)\}.$$

$$\text{FOC w.r.t. } c: \quad \frac{\partial V(k)}{\partial c} = u'(c) - \lambda_t = 0.$$

$$\text{FOC w.r.t. } \tilde{k}: \quad \frac{\partial V(k)}{\partial \tilde{k}} = \beta \frac{\partial V(\tilde{k})}{\partial \tilde{k}} - \lambda_t = 0.$$

Now we can substitute out the Lagrange multiplier from the first FOC into the second one to obtain:

$$\beta \frac{\partial V(\tilde{k})}{\partial \tilde{k}} = u'(c). \tag{2}$$

Consumption and Savings: Solution without Substitution II

To eliminate the partial of the value function, we use the Envelope Theorem as usual:

$$ET: \quad \frac{\partial V(k)}{\partial k} = \lambda_t (f'(k) + 1 - \delta).$$

First, again substitute out the multiplier from the first FOC:

$$\frac{\partial V(k)}{\partial k} = u'(c)(f'(k) + 1 - \delta).$$

And finally substitute after shifting forward in the equation (2):

$$u'(c) = u'(\tilde{c})\beta(f'(\tilde{k}) + 1 - \delta).$$

This is the standard EE obtained in the substitution way.