Macro Topics: Introduction to Matlab
Fall 2016
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Homework #6

Suggested Solutions

Problem 1. *Consider the following problem of a Central Bank which tries to minimize the following loss function:*

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \left[(\pi_t - \pi^*)^2 + \lambda (l_t - l^*)^2 \right],$$

subject to

$$l_t = \alpha \pi_t + \rho l_{t-1} + C, \quad l_{-1} \text{ given},$$

where π_t is the inflation rate, l_t is the employment rate, $\beta \in (0,1)$, $\lambda > 0$, $\alpha > 0$, $\rho \in (0,1)$. The Central Bank can directly control inflation rate π_t.

(1) Clearly identify state and control variables. Write the problem in a standard LQ form. (Hint: Problem has more than one way of solving. For one of them you can consider unity as a state variable.)
(2) Guessing that the value function is quadratic in the state variable, use guess–and–verify method to derive the optimal policy and the value function of the Central Bank. (Hint: You can use the formulae derived in the class.)
(3) Solve the problem using standard procedure (FOC, ET, EE).

Solution:

(1) In order to simplify the problem we will take unity as a state variable. Since Central Bank can control inflation, we will take π_t as control variable, and l_{t-1} and 1 as state variables. Denote by x_t current state variable and by u_t control variable:

$$x_t = \begin{pmatrix} l_{t-1} \\ 1 \end{pmatrix}, \quad u_t = \pi_t.$$

We start by rewriting constraint as

$$x_{t+1} = A x_t + B u_t,$$

where A and B are some matrices. We have:

$$\underbrace{\begin{pmatrix} l_t \\ 1 \end{pmatrix}}_{x_{t+1}} = \underbrace{\begin{pmatrix} \rho & C \\ 0 & 1 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} l_{t-1} \\ 1 \end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix} \alpha \\ 0 \end{pmatrix}}_{B} \underbrace{\pi_t}_{u_t}.$$

Note that we added extra equation $1 = 1$ as an additional constraint for state variable. Now we can rewrite objective function in the form:

$$x_t' R x_t + u_t' Q u_t + 2 u_t' H x_t.$$

We have

$$(\pi_t - \pi^*)^2 + \lambda(l_t - l^*)^2 = (\pi_t - \pi^*)^2 + \lambda(\alpha\pi_t + \rho l_{t-1} + C - l^*)^2 =$$

$$= \pi_t^2 - 2\pi_t\pi^* + (\pi^*)^2 + \lambda\alpha^2\pi_t^2 + \lambda\rho^2 l_{t-1}^2 + \lambda(C-l^*)^2 + 2\lambda\alpha\rho\pi_t l_{t-1} + 2\lambda\alpha(C-l^*)\pi_t + 2\lambda\rho(C-l^*)l_{t-1} =$$

$$= x_t' \underbrace{\begin{pmatrix} \lambda\rho^2 & \lambda\rho(C-l^*) \\ \lambda\rho(C-l^*) & (\pi^*)^2 + \lambda(C-l^*)^2 \end{pmatrix}}_{-R} x_t + u_t' \underbrace{(1+\lambda\alpha^2)}_{-Q} u_t + 2u_t' \underbrace{\begin{pmatrix} \lambda\alpha\rho & \lambda\alpha(C-l^*) - \pi^* \end{pmatrix}}_{-H} x_t.$$

(2) Now we make a guess that our value function is quadratic in state variable, that is:

$$V(x_t) = x_t' P x_t,$$

where matrix P is of the form

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}.$$

Using guess–and–verify method we need to find coefficients p_{ij}, $i, j \in \{1, 2\}$. We start by writing the Bellman equation:

$$V(x_t) = \max\left\{-x_t' R x_t - u_t' Q u_t - 2u_t' H x_t + \beta V(x_{t+1})\right\},$$

or substituting for V and for x_{t+1} we get:

$$x_t' P x_t = \max\left\{-x_t' R x_t - u_t' Q u_t - 2u_t' H x_t + \beta(Ax_t + Bu_t)' P (Ax_t + Bu_t)\right\}.$$

In the class we have already derived the following formula

$$u_t = -(Q + \beta B'PB)^{-1}(\beta B'PA + H)x_t,$$

where matrix P solves

$$P = R + \beta A'PA - (\beta A'PB + H')(Q + \beta B'PB)^{-1}(\beta B'PA + H).$$

Important note: You should know how to get this equation. For details please see the suggested solutions for the last exercise sessions.
We start by calculating

$$R + \beta A'PA = \begin{pmatrix} -\lambda\rho^2 & -\lambda\rho(C-l^*) \\ -\lambda\rho(C-l^*) & -(\pi^*)^2 - \lambda(C-l^*)^2 \end{pmatrix} + \beta \begin{pmatrix} \rho & C \\ 0 & 1 \end{pmatrix}' \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \rho & C \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} -\lambda\rho^2 + \beta\rho^2 p_{11} & -\lambda\rho(C-l^*) + \beta(\rho C p_{11} + \rho p_{12}) \\ -\lambda\rho(C-l^*) + \beta(\rho C p_{11} + \rho p_{21}) & -(\pi^*)^2 - \lambda(C-l^*)^2 + \beta(C^2 p_{11} + C p_{21} + C p_{12} + p_{22}) \end{pmatrix}.$$

Now we need to calculate

$$(\beta A'PB + H')(Q + \beta B'PB)^{-1}(\beta B'PA + H),$$

and we will calculate each part separately. Thus we have

$$\beta A'PB + H' = \beta \begin{pmatrix} \rho & C \\ 0 & 1 \end{pmatrix}' \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} -\lambda\alpha\rho \\ -\lambda\alpha(C-l^*) + \pi^* \end{pmatrix} =$$

$$= \begin{pmatrix} -\lambda\alpha\rho + \beta\alpha\rho p_{11} \\ -\lambda\alpha(C - l^*) + \pi^* + \beta C\alpha p_{11} + \beta\alpha p_{21} \end{pmatrix},$$

$$Q + \beta B'PB = -1 - \lambda\alpha^2 + \beta \begin{pmatrix} \alpha \\ 0 \end{pmatrix}' \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \begin{pmatrix} \alpha \\ 0 \end{pmatrix} = -1 - \lambda\alpha^2 + \beta\alpha^2 p_{11},$$

$$\beta B'PA + H = (\beta A'PB + H')' = \begin{pmatrix} -\lambda\alpha\rho + \beta\alpha\rho p_{11} \\ -\lambda\alpha(C - l^*) + \pi^* + \beta C\alpha p_{11} + \beta\alpha p_{21} \end{pmatrix}'.$$

Now we have

$$(\beta A'PB + H')(Q + \beta B'PB)^{-1}(\beta B'PA + H) =$$

$$= \begin{pmatrix} \frac{(-\lambda\alpha\rho+\beta\alpha\rho p_{11})^2}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} & \frac{(-\lambda\alpha\rho+\beta\alpha\rho p_{11})(-\lambda\alpha(C-l^*)+\pi^*+\beta C\alpha p_{11}+\beta\alpha p_{21})}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} \\ \frac{(-\lambda\alpha\rho+\beta\alpha\rho p_{11})(-\lambda\alpha(C-l^*)+\pi^*+\beta C\alpha p_{11}+\beta\alpha p_{21})}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} & \frac{(-\lambda\alpha(C-l^*)+\pi^*+\beta C\alpha p_{11}+\beta\alpha p_{21})^2}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} \end{pmatrix}.$$

Therefore we need to solve equation of the form

$$\begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} = \begin{pmatrix} -\lambda\rho^2 + \beta\rho^2 p_{11} & -\lambda\rho(C - l^*) + \beta(\rho C p_{11} + \rho p_{12}) \\ -\lambda\rho(C - l^*) + \beta(\rho C p_{11} + \rho p_{21}) & -(\pi^*)^2 - \lambda(C - l^*)^2 + \beta(C^2 p_{11} + C p_{21} + C p_{12} + p_{22}) \end{pmatrix} -$$

$$- \begin{pmatrix} \frac{(-\lambda\alpha\rho+\beta\alpha\rho p_{11})^2}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} & \frac{(-\lambda\alpha\rho+\beta\alpha\rho p_{11})(-\lambda\alpha(C-l^*)+\pi^*+\beta C\alpha p_{11}+\beta\alpha p_{21})}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} \\ \frac{(-\lambda\alpha\rho+\beta\alpha\rho p_{11})(-\lambda\alpha(C-l^*)+\pi^*+\beta C\alpha p_{11}+\beta\alpha p_{21})}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} & \frac{(-\lambda\alpha(C-l^*)+\pi^*+\beta C\alpha p_{11}+\beta\alpha p_{21})^2}{-1-\lambda\alpha^2+\beta\alpha^2 p_{11}} \end{pmatrix}.$$

(3) Another way to solve the problem is simple Bellman equation approach. We need to solve the following problem

$$\max - \sum_{t=0}^{\infty} \beta^t \left[(\pi_t - \pi^*)^2 + \lambda(l_t - l^*)^2 \right],$$

subject to

$$l_t = \alpha\pi_t + \rho l_{t-1} + C.$$

We take l_{t-1} as a state variable and l_t as control variable (we will eliminate π_t using the constraint). Therefore the Bellman equation is:

$$V(l_{t-1}) = \max \left\{ -\frac{(l_t - \rho l_{t-1} - \pi^*\alpha - C)^2}{\alpha^2} - \lambda(l_t - l^*)^2 + \beta V(l_t) \right\}.$$

FOC (derivative with respect to l_t) is:

$$-\frac{2(l_t - \rho l_{t-1} - \pi^*\alpha - C)}{\alpha^2} - 2\lambda(l_t - l^*) + \beta V'(l_t) = 0.$$

ET condition (derivative with respect to l_{t-1}) is:

$$V'(l_{t-1}) = \frac{2\rho(l_t - \rho l_{t-1} - \pi^*\alpha - C)}{\alpha^2}.$$

We shift ET condition one period forward and plug it into FOC to get the Euler Equation:

$$V'(l_t) = \frac{2\rho(l_{t+1} - \rho l_t - \pi^*\alpha - C)}{\alpha^2},$$

$$-\frac{2(l_t - \rho l_{t-1} - \pi^*\alpha - C)}{\alpha^2} - 2\lambda(l_t - l^*) + \frac{2\beta\rho(l_{t+1} - \rho l_t - \pi^*\alpha - C)}{\alpha^2} = 0.$$