Macro Topics: Introduction to Matlab
Fall 2016
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Homework #5

Suggested Solutions

Problem 1. *Consider social planner that seeks to maximize lifetime utility of a representative agent:*

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$$

subject to

$$c_t + \frac{b_{t+1}}{R} + k_{t+1} = w_t(1-\tau)l_t + r_t k_t + (1-\delta)k_t + b_t, \quad R > 0,$$

where c_t is consumption, b_t is money holdings (bonds), k_t is capital, l_t is labour input. Every period household allocates its resources between consumption, money and capital. Household receives wage w_t (taxed by rate τ) from work, and rent from capital r_t it allocates to firms. Household takes tax τ, wage w_t, and rent r_t as given.

There is a continuum of firms, that produce single output using Cobb–Douglas production function of the form:

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha},$$

where $A_t = A_{t-1}^{\rho} v_t$, $\mathbb{E}\log(v_t) = 0$. Firms pay wage and capital rent to the household. Market for bonds clears.

1. *Clearly identify state and control variables (pay attention to all given conditions).*
2. *Write down the Bellman equation for the problem (that is, write the problem in a recursive form).*
3. *Derive First Order Conditions and Envelope Theorem Conditions.*
4. *Derive the Euler Equation for the problem.*
5. *Solve firm's maximization problem to get the wage and the capital rent. (Hint: Set wage and rent equal to marginal product of labour and capital respectively.)*
6. *Assume that utility function is of the form:*

   $$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{l_t^{1+\phi}}{1+\phi}.$$

   Rewrite Euler Equation, FOCs, and resource constraint using the above functional form for u and firm's FOC (derived equations for w_t and r_t).
7. *Log–linearize the above equations using Taylor series approximation.*
8. *Use Uhlig's method to log–linearize equations from (6). Compare two methods, comment.*

Solution:

(1) Since market for bonds clears in the representative agent framework we have that $\forall t$ $b_t = 0$. Therefore we can rewrite budget constraint as

$$c_t + k_{t+1} = w_t(1-\tau)l_t + r_t k_t + (1-\delta)k_t,$$

or solving for c_t we get

$$c_t = w_t(1-\tau)l_t + (1-\delta+r_t)k_t - k_{t+1}.$$

We take k_t and A_t as state variables and c_t, l_t, k_{t+1}, and y_t as controls.

(2) The Bellman equation is (we will keep c_t in the equation for now, but we actually eliminate it using the above equation):

$$V(k_t, A_t) = \max_{k_{t+1}, l_t} \{u(c_t, l_t) + \beta \mathbb{E}_t V(k_{t+1}, A_{t+1})\},$$

where $c_t = w_t(1-\tau)l_t + (1-\delta+r_t)k_t - k_{t+1}$.

(3) FOCs (derivatives with respect to k_{t+1} and l_t) are:

$$-u_1(c_t, l_t) + \beta \mathbb{E}_t V_1(k_{t+1}, A_{t+1}) = 0,$$

$$w_t(1-\tau)u_1(c_t, l_t) + u_2(c_t, l_t) = 0.$$

ET condition (derivative with respect to k_t only, A_t is predetermined state) is:

$$V_1(k_t, A_t) = u_1(c_t, l_t)(1-\delta+r_t).$$

(4) To get the Euler Equation we shift ET condition one period ahead

$$V_1(k_{t+1}, A_{t+1}) = u_1(c_{t+1}, l_{t+1})(1-\delta+r_{t+1})$$

and substitute it to the first FOC:

$$u_1(c_t, l_t) = \beta \mathbb{E}_t[u_1(c_{t+1}, l_{t+1})(1-\delta+r_{t+1})].$$

(5) Since there are many firms, they act competitively and set wage and rent equal to marginal product of labour and capital respectively. Therefore,

$$w_t = \frac{\partial y_t}{\partial l_t} = (1-\alpha)A_t k_t^{\alpha} l_t^{-\alpha},$$

$$r_t = \frac{\partial y_t}{\partial k_t} = \alpha A_t k_t^{\alpha-1} l_t^{1-\alpha}.$$

(6) Using the fact that

$$u(c_t, l_t) = \frac{c_t^{1-\sigma}}{1-\sigma} - \frac{l_t^{1+\phi}}{1+\phi}$$

and the above equations we get:

$$c_t^{-\sigma} = \beta \mathbb{E}_t[c_{t+1}^{-\sigma}(1-\delta+\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha})], \tag{1}$$

$$(1-\tau)(1-\alpha)A_t k_t^{\alpha} l_t^{-\alpha} c_t^{-\sigma} = l_t^{\phi}, \tag{2}$$

$$c_t = ((1-\alpha)(1-\tau)+\alpha)A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t - k_{t+1}, \tag{3}$$

$$A_t = A_{t-1}^{\rho} v_t. \tag{4}$$

(7) Now we can proceed with log–linearization of the above equations. Let's start with equation (1). We have

$$c_t^{-\sigma} = \beta \mathbb{E}_t[c_{t+1}^{-\sigma}(1 - \delta + \alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha})].$$

In the steady state the equation becomes

$$\bar{c}^{-\sigma} = \beta \bar{c}^{-\sigma}(1 - \delta + \alpha \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha}).$$

Subtracting this expression from both sides of the equation above we get

$$c_t^{-\sigma} - \bar{c}^{-\sigma} = \beta \mathbb{E}_t[(c_{t+1}^{-\sigma} - \bar{c}^{-\sigma})(1 - \delta) + \alpha(A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} c_{t+1}^{-\sigma} - \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \bar{c}^{-\sigma})].$$

We start by linearizing left side of the equation. Using Taylor series approximation around steady–state we get:

$$c_t^{-\sigma} = \bar{c}^{-\sigma} - \sigma \bar{c}^{-\sigma-1}(c_t - \bar{c}) = \bar{c}^{-\sigma} - \sigma \bar{c}^{-\sigma} \tilde{c}_t,$$

where

$$\tilde{c}_t = \frac{c_t - \bar{c}}{\bar{c}}.$$

Now we need to linearize the right side of our equation. Again, using Taylor series approximation around steady–state we have:

$$A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} c_{t+1}^{-\sigma} = \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \bar{c}^{-\sigma} + \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \bar{c}^{-\sigma} \bar{A} \tilde{A}_{t+1} + (\alpha - 1) \bar{A} \bar{k}^{\alpha-2} \bar{l}^{1-\alpha} \bar{c}^{-\sigma} \bar{k} \tilde{k}_{t+1} +$$

$$+ (1 - \alpha) \bar{A} \bar{k}^{\alpha-1} \bar{l}^{-\alpha} \bar{c}^{-\sigma} \bar{l} \tilde{l}_{t+1} + (-\sigma) \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \bar{c}^{-\sigma-1} \bar{c} \tilde{c}_{t+1} =$$

$$= \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \bar{c}^{-\sigma} + \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \bar{c}^{-\sigma} (\tilde{A}_{t+1} - (1 - \alpha) \tilde{k}_{t+1} + (1 - \alpha) \tilde{l}_{t+1} - \sigma \tilde{c}_{t+1}).$$

Substituting everything to the original equation we get

$$-\sigma \bar{c}^{-\sigma} \tilde{c}_t = \beta \mathbb{E}_t[-\sigma(1-\delta) \bar{c}^{-\sigma} \tilde{c}_{t+1} + \alpha \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} \bar{c}^{-\sigma} (\tilde{A}_{t+1} - (1-\alpha) \tilde{k}_{t+1} + (1-\alpha) \tilde{l}_{t+1} - \sigma \tilde{c}_{t+1})] \Rightarrow$$

$$\Rightarrow \sigma \tilde{c}_t = \beta \sigma (1-\delta) \mathbb{E}_t \tilde{c}_{t+1} - \beta \alpha \bar{A} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha} (\mathbb{E}_t \tilde{A}_{t+1} - (1-\alpha) \mathbb{E}_t \tilde{k}_{t+1} + (1-\alpha) \mathbb{E}_t \tilde{l}_{t+1} - \sigma \mathbb{E}_t \tilde{c}_{t+1}),$$

which is linear.

Now we need to linearize equation (2)

$$(1 - \tau)(1 - \alpha) A_t k_t^{\alpha} l_t^{-\alpha} c_t^{-\sigma} = l_t^{\phi}.$$

Again, in the steady–state we get:

$$(1 - \tau)(1 - \alpha) \bar{A} \bar{k}^{\alpha} \bar{l}^{-\alpha} \bar{c}^{-\sigma} = \bar{l}^{\phi}.$$

Subtracting this expression from both sides of the equation above we get

$$(1 - \tau)(1 - \alpha)(A_t k_t^{\alpha} l_t^{-\alpha} c_t^{-\sigma} - \bar{A} \bar{k}^{\alpha} \bar{l}^{-\alpha} \bar{c}^{-\sigma}) = l_t^{\phi} - \bar{l}^{\phi}.$$

Right side of the equation can be approximated as:

$$l_t^{\phi} = \bar{l}^{\phi} + \phi \bar{l}^{\phi-1} \bar{l} \tilde{l}_t = \bar{l}^{\phi} + \phi \bar{l}^{\phi} \tilde{l}_t,$$

and the left side after the approximation becomes

$$A_t k_t^{\alpha} l_t^{-\alpha} c_t^{-\sigma} = \bar{A} \bar{k}^{\alpha} \bar{l}^{-\alpha} \bar{c}^{-\sigma} + \bar{A} \bar{k}^{\alpha} \bar{l}^{-\alpha} \bar{c}^{-\sigma} (\tilde{A}_t + \alpha \tilde{k}_t - \alpha \tilde{l}_t - \sigma \tilde{c}_t).$$

Therefore, we can rewrite original equation as

$$(1-\tau)(1-\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{-\alpha}\bar{c}^{-\sigma}(\tilde{A}_t+\alpha\tilde{k}_t-\alpha\tilde{l}_t-\sigma\tilde{c}_t)=\phi\bar{l}^{\phi}\tilde{l}_t.$$

Now we need to linearize the budget constraint (3)

$$c_t=((1-\alpha)(1-\tau)+\alpha)A_tk_t^{\alpha}l_t^{1-\alpha}+(1-\delta)k_t-k_{t+1}.$$

In the steady–state we have

$$\bar{c}=((1-\alpha)(1-\tau)+\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}+(1-\delta)\bar{k}-\bar{k},$$

and thus

$$c_t-\bar{c}=((1-\alpha)(1-\tau)+\alpha)(A_tk_t^{\alpha}l_t^{1-\alpha}-\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha})+(1-\delta)(k_t-\bar{k})-(k_{t+1}-\bar{k}).$$

The only non–linear term is in the right side of the equation, and so using Taylor series approximation around the steady–state we have:

$$A_tk_t^{\alpha}l_t^{1-\alpha}=\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}+\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}(\tilde{A}_t+\alpha\tilde{k}_t+(1-\alpha)\tilde{l}_t).$$

Thus we have

$$\bar{c}\tilde{c}_t=((1-\alpha)(1-\tau)+\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}(\tilde{A}_t+\alpha\tilde{k}_t+(1-\alpha)\tilde{l}_t)+(1-\delta)\bar{k}\tilde{k}_t-\bar{k}\tilde{k}_{t+1}.$$

We are left with equation (4)

$$A_t=A_{t-1}^{\rho}v_t.$$

In this case it is better to rewrite it as

$$\log A_t=\rho\log A_{t-1}+\log v_t.$$

Since

$$\tilde{A}_t=\log\frac{A_t}{\bar{A}}=\log A_t-\log\bar{A}$$

above equation becomes

$$\tilde{A}_t+\log\bar{A}=\rho(\tilde{A}_{t-1}+\log\bar{A})+\tilde{v}_t+\log\bar{v}.$$

In the steady–state we get:

$$\bar{A}=\bar{A}^{\rho}\bar{v}\quad\Rightarrow\quad\log\bar{A}=\rho\log\bar{A}+\log\bar{v}.$$

Therefore, we can rewrite original equation as

$$\tilde{A}_t=\rho\tilde{A}_{t-1}+\tilde{v}_t.$$

(8) $c_t=\bar{c}e^{\tilde{c}_t}$, $A_t=\bar{A}e^{\tilde{A}_t}$, $k_t=\bar{k}e^{\tilde{k}_t}$, $l_t=\bar{l}e^{\tilde{l}_t}$, $v_t=\bar{v}e^{\tilde{v}_t}$.
We start with equation (1):

$$\bar{c}^{-\sigma}e^{-\sigma\tilde{c}_t}=\beta\mathbb{E}_t[\bar{c}^{-\sigma}e^{-\sigma\tilde{c}_{t+1}}(1-\delta+\alpha\bar{A}\bar{k}^{\alpha-1}\bar{l}^{1-\alpha}e^{\tilde{A}_{t+1}-(1-\alpha)\tilde{k}_{t+1}+(1-\alpha)\tilde{l}_{t+1}})]\ \Rightarrow$$

$$\Rightarrow\ e^{-\sigma\tilde{c}_t}=\beta\mathbb{E}_t[(1-\delta)e^{-\sigma\tilde{c}_{t+1}}+\alpha\bar{A}\bar{k}^{\alpha-1}\bar{l}^{1-\alpha}e^{\tilde{A}_{t+1}-(1-\alpha)\tilde{k}_{t+1}+(1-\alpha)\tilde{l}_{t+1}-\sigma\tilde{c}_{t+1}}].$$

Using approximation $e^x \approx 1+x$, we get:

$1-\sigma\tilde{c}_t = \beta\mathbb{E}_t[(1-\delta)(1-\sigma\tilde{c}_{t+1})+\alpha\bar{A}\bar{k}^{\alpha-1}\bar{l}^{1-\alpha}(1+\tilde{A}_{t+1}-(1-\alpha)\tilde{k}_{t+1}+(1-\alpha)\tilde{l}_{t+1}-\sigma\tilde{c}_{t+1})]$.

Using EE in steady states:

$$1 = \beta(1-\delta+\alpha\bar{A}\bar{k}^{\alpha-1}\bar{l}^{1-\alpha}),$$

we get

$\sigma\tilde{c}_t = \beta\sigma(1-\delta)\mathbb{E}_t\tilde{c}_{t+1} - \beta\alpha\bar{A}\bar{k}^{\alpha-1}\bar{l}^{1-\alpha}(\mathbb{E}_t\tilde{A}_{t+1}-(1-\alpha)\mathbb{E}_t\tilde{k}_{t+1}+(1-\alpha)\mathbb{E}_t\tilde{l}_{t+1}-\sigma\mathbb{E}_t\tilde{c}_{t+1})$,

which is linear.

Now we proceed with equation (2):

$$(1-\tau)(1-\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{-\alpha}\bar{c}^{-\sigma}e^{\tilde{A}_t+\alpha\tilde{k}_t-\alpha\tilde{l}_t-\sigma\tilde{c}_t} = \bar{l}^{\phi}e^{\phi\tilde{l}_t}.$$

Using approximation $e^x \approx 1+x$, we get:

$$(1-\tau)(1-\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{-\alpha}\bar{c}^{-\sigma}(1+\tilde{A}_t+\alpha\tilde{k}_t-\alpha\tilde{l}_t-\sigma\tilde{c}_t) = \bar{l}^{\phi}(1+\phi\tilde{l}_t).$$

Using equation (2) in steady states

$$(1-\tau)(1-\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{-\alpha}\bar{c}^{-\sigma} = \bar{l}^{\phi},$$

we get

$$(1-\tau)(1-\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{-\alpha}\bar{c}^{-\sigma}(\tilde{A}_t+\alpha\tilde{k}_t-\alpha\tilde{l}_t-\sigma\tilde{c}_t) = \phi\bar{l}^{\phi}\tilde{l}_t.$$

Now we proceed with equation (3):

$$\bar{c}e^{\tilde{c}_t} = ((1-\alpha)(1-\tau)+\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}e^{\tilde{A}_t+\alpha\tilde{k}_t+(1-\alpha)\tilde{l}_t} + (1-\delta)\bar{k}e^{\tilde{k}_t} - \bar{k}e^{\tilde{k}_{t+1}}.$$

Using approximation $e^x \approx 1+x$, we get:

$\bar{c}(1+\tilde{c}_t) = ((1-\alpha)(1-\tau)+\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}(1+\tilde{A}_t+\alpha\tilde{k}_t+(1-\alpha)\tilde{l}_t)+(1-\delta)\bar{k}(1+\tilde{k}_t)-\bar{k}(1+\tilde{k}_{t+1})$.

Using equation (3) in steady states

$$\bar{c} = ((1-\alpha)(1-\tau)+\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha} + (1-\delta)\bar{k} - \bar{k},$$

we get

$$\bar{c}\tilde{c}_t = ((1-\alpha)(1-\tau)+\alpha)\bar{A}\bar{k}^{\alpha}\bar{l}^{1-\alpha}(\tilde{A}_t+\alpha\tilde{k}_t+(1-\alpha)\tilde{l}_t) + (1-\delta)\bar{k}\tilde{k}_t - \bar{k}\tilde{k}_{t+1}.$$

Now we proceed with equation (4):

$$\bar{A}e^{\tilde{A}_t} = \bar{A}^{\rho}\bar{v}e^{\rho\tilde{A}_{t-1}+\tilde{v}_t}.$$

Using approximation $e^x \approx 1+x$, we get:

$$\bar{A}(1+\tilde{A}_t) = \bar{A}^{\rho}\bar{v}(1+\rho\tilde{A}_{t-1}+\tilde{v}_t).$$

Using equation (4) in steady states

$$\bar{A} = \bar{A}^{\rho}\bar{v},$$

we get

$$\tilde{A}_t = \rho\tilde{A}_{t-1}+\tilde{v}_t.$$

As we can see, we receive exactly the same equations as in part (7).