Macro Topics: Introduction to Matlab
Fall 2016
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Homework #4

Suggested Solutions

Problem 1. ***(A model of quitting)***

Consider the following problem of a worker who decides when to quit his job. He wants to maximize

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t y_t, \quad 0 < \beta < 1,$$

where y_t is his income in period t.

Each period the worker receives a new job offer, irrespective of his employment history. New jobs are identical. They pay wage $w_0 > 0$ and are available to the worker without having to search. Subsequent wages on any job evolve as follows. With probability $\phi > 0$ the wage remains the same as it was last period and with probability $(1-\phi)$ it is drawn from a fixed distribution with c.d.f. F with $F(0) = 0$ and $F(B) = 1$ for some $B > w_0$ and finite. The worker observes his current wage and decides whether to quit his current job and start a new one.

(a) Formulate the Bellman equation for the worker's problem.

(b) Show that the worker's optimal policy is characterized by a reservation wage, and solve for it. (Hint: Assume that $V(\cdot)$ is strictly monotone.)

Suppose now that the worker can only receive the starting wage w_0 by searching. The objective function becomes

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t (y_t - \gamma(s_t)),$$

where $\gamma(s)$ is the cost of searching with intensity s. Assume $\gamma'(s) > 0$. The worker can only search for a job in those periods when he is unemployed. If he searches with intensity s then at the beginning of the following period, with probability $\pi(s)$, he receives an offer to work at the wage w_0, where π is differentiable, strictly increasing, and strictly concave function. The worker remains unemployed with probability $(1-\pi(s))$. Subsequent wages on any job evolve as before. Finally, assume that if the worker quits his job, he must pay a job destruction tax $\tau > 0$ to the government.

(c) Formulate the Bellman equation(s) for the worker's problem. (Hint: write down two Bellman equations: one for $V(0)$ (value of being unemployed and searching a job with intensity s) and another for $V(w)$.)

(d) Show that the optimal policy again has a "reservation wage" characterization. (Hint: It's enough to derive $\bar{w}$ as a function of $\int_0^B V(w')dF(w')$.)

(e) What is the effect on the reservation wage of an increase in the tax on job destruction?

Solution:

(a) $$V(w) = \max\Big\{ \underbrace{w + \phi\beta V(w) + (1-\phi)\beta \int_0^B V(w')dF(w')}_{\text{keep current job with current wage } w},$$

$$\underbrace{w_0 + \phi\beta V(w_0) + (1-\phi)\beta \int_0^B V(w')dF(w')}_{\text{leave current job and get a new one with current wage } w_0} \Big\}.$$

(b) There is a reservation wage $\bar{w}$ which makes a worker indifferent beatween leaving a job or keeping it.

$$\bar{w} + \phi\beta V(\bar{w}) + (1-\phi)\beta \int_0^B V(w')dF(w') = w_0 + \phi\beta V(w_0) + (1-\phi)\beta \int_0^B V(w')dF(w'),$$

$$\bar{w} + \phi\beta V(\bar{w}) = w_0 + \phi\beta V(w_0).$$

If we assume that $V(\cdot)$ is strictly monotone, we get $\bar{w} = w_0$.

(c) $V(0) = \max_s \{-\gamma(s) + \pi(s)\beta V(w_0) + (1-\pi(s))\beta V(0)\}$,

$V(w) = \max\left\{ w + \phi\beta V(w) + (1-\phi)\beta \int_0^B V(w')dF(w'), -\tau + V(0) \right\}$.

(d) FOC: $[s]$: $-\gamma'(s^*) + \beta\pi'(s^*)V(w_0) - \beta\pi'(s^*)V(0) = 0 \Rightarrow V(0) = V(w_0) - \dfrac{\gamma'(s^*)}{\beta\pi'(s^*)}$.

$$V(w) = \max\left\{ w + \phi\beta V(w) + (1-\phi)\beta \int_0^B V(w')dF(w'), -\tau + V(w_0) - \frac{\gamma'(s^*)}{\beta\pi'(s^*)} \right\}.$$

There is a reservation wage $\bar{w}$ which makes a worker indifferent beatween leaving a job or keeping it:

$$\bar{w} + \phi\beta V(\bar{w}) + (1-\phi)\beta \int_0^B V(w')dF(w') = -\tau + V(w_0) - \frac{\gamma'(s^*)}{\beta\pi'(s^*)} = V(\bar{w}),$$

$$\bar{w} + (1-\phi)\beta \int_0^B V(w')dF(w') = (1-\phi\beta)V(\bar{w}),$$

$$V(\bar{w}) = \frac{\bar{w}}{1-\phi\beta} + \frac{(1-\phi)\beta}{1-\phi\beta} \int_0^B V(w')dF(w'),$$

$$\bar{w} + \frac{\phi\beta}{1-\phi\beta}\bar{w} + \frac{\phi(1-\phi)\beta^2}{1-\phi\beta} \int_0^B V(w')dF(w') + (1-\phi)\beta \int_0^B V(w')dF(w') = V(w_0) - \tau - \frac{\gamma'(s^*)}{\beta\pi'(s^*)},$$

$$\frac{\bar{w}}{1-\phi\beta} + \frac{(1-\phi)\beta}{1-\phi\beta}\int_0^B V(w')dF(w') = V(w_0) - \tau - \frac{\gamma'(s^*)}{\beta\pi'(s^*)},$$

$$\bar{w} = (1-\phi\beta)\left(V(w_0) - \tau - \frac{\gamma'(s^*)}{\beta\pi'(s^*)}\right) - (1-\phi)\beta\int_0^B V(w')dF(w').$$

(e) $\dfrac{\partial\bar{w}}{\partial\tau} = -(1-\phi\beta) < 0 \Rightarrow$ the effect is negative.

Problem 2. ***(LS 6.3: A random number of offers per period)***

An unemployed worker is confronted with a random number, n, of job offers each period. With probability π_n, the worker receives n offers in a given period, where $\pi_n \geq 0$ for $n \geq 1$, and $\sum_{n=1}^{N} \pi_n = 1$ for $N < +\infty$. Each offer is drawn independently from the same distribution $F(w)$. Assume that the number of offers n is independently distributed across time. The worker works forever at wage w after having accepted a job and receives unemployment compensation of c during each period of unemployment. He chooses a strategy to maximize $\mathbb{E}\sum_{t=0}^{\infty} \beta^t y_t$, where $y_t = c$ if he is unemployed, $y_t = w$ if he is employed.

Let $V(w)$ be the value of the objective function of an unemployed worker who has best offer w in hand and who proceeds optimally. Formulate the Bellman equation for this worker.

Solution:

$$V(w) = \max\left\{\frac{w}{1-\beta},\ c + \sum_{n=1}^{N} \pi_n \beta \int V(w')d(F^n)(w')\right\}.$$

In effect, the worker is confronted with a lottery with probabilities π_n over distributions $F^n(w)$, from which he will sample next period. Here w is the highest offer in hand.