Macro Topics: Introduction to Matlab
Fall 2016
Instructor: Olga Bychkova

Homework #3

Suggested Solutions

Problem 1. *Consider the following habit persistence problem:*

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\log c_t + \gamma \log c_{t-1})$$

subject to the following constraints:

$$k_{t+1} + c_t \leq A k_t^{\alpha}, \quad A > 0, \ \ \alpha \in (0,1),$$

$$c_t > 0, \ \ k_t > 0, \ \ k_0, c_{-1} \text{ given.}$$

Guessing that the value function is of the form:

$$V(k_t, c_{t-1}) = E + F \log k_t + G \log c_{t-1},$$

derive constants E, F, and G. Calculate the optimal policy given your value function.

Solution: We eliminate one control using the equation

$$c_t = A k_t^{\alpha} - k_{t+1},$$

and Bellman equation becomes

$$V(k, c_{-1}) = \max_{\tilde{k}} \left\{ \log(Ak^{\alpha} - \tilde{k}) + \gamma \log c_{-1} + \beta V(\tilde{k}, Ak^{\alpha} - \tilde{k}) \right\},$$

where $k = k_t$, $\tilde{k} = k_{t+1}$, $c_{-1} = c_{t-1}$. We make a guess that

$$V(k, c_{-1}) = E + F \log k + G \log c_{-1}.$$

Now substituting for V into Bellman equation we get

$$E + F \log k + G \log c_{-1} = \max_{\tilde{k}} \left\{ \log(Ak^{\alpha} - \tilde{k}) + \gamma \log c_{-1} + \beta \left(E + F \log \tilde{k} + G \log(Ak^{\alpha} - \tilde{k}) \right) \right\}.$$

The FOC (derivative with respect to $\tilde{k}$) is:

$$-\frac{1}{Ak^{\alpha} - \tilde{k}} + \beta F \frac{1}{\tilde{k}} - \beta G \frac{1}{Ak^{\alpha} - \tilde{k}} = 0 \ \ \Rightarrow \ \ \tilde{k} = \frac{\beta F A k^{\alpha}}{\beta(F+G) + 1}.$$

ET condition with respect to c_{-1} is

$$\frac{G}{c_{-1}} = \frac{\gamma}{c_{-1}} \ \ \Rightarrow \ \ G = \gamma.$$

ET condition with respect to k is

$$\frac{F}{k} = \frac{\alpha A k^{\alpha-1}}{A k^{\alpha} - \tilde{k}} + \beta G \frac{\alpha A k^{\alpha-1}}{A k^{\alpha} - \tilde{k}},$$

$$\frac{F}{k} = \frac{\alpha A k^{\alpha-1}}{A k^{\alpha} - \beta F A k^{\alpha}/(\beta(F+G)+1)} + \beta G \frac{\alpha A k^{\alpha-1}}{A k^{\alpha} - \beta F A k^{\alpha}/(\beta(F+G)+1)},$$

$$\frac{F}{k} = \frac{\alpha A k^{\alpha-1}}{A k^{\alpha} - \beta F A k^{\alpha}/(\beta(F+\gamma)+1)} + \beta\gamma \frac{\alpha A k^{\alpha-1}}{A k^{\alpha} - \beta F A k^{\alpha}/(\beta(F+\gamma)+1)} \quad \Rightarrow \quad F = \frac{\alpha(1+\beta\gamma)}{1-\alpha\beta}.$$

Substituting for G, F, and $\tilde{k}$ into Bellman equation and solving for E, we get:

$$E = \frac{1}{1-\beta}\left(\frac{1+\beta\gamma}{1-\alpha\beta}\log(\alpha\beta A) - (1-\beta\gamma)\log\left(\frac{\alpha\beta}{1-\alpha\beta}\right)\right).$$

Also we can solve for $\tilde{k}$

$$\tilde{k} = \alpha\beta A k^{\alpha}.$$

Problem 2. *Consider the standard Robinson Crusoe problem:*

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \quad c_t = A k_t^{\alpha} + (1-\delta)k_t - k_{t+1}.$$

(1) Set parameters in the following way: $A = 2$, $\alpha = 1/3$, $\beta = 0.98$, $\delta = 1$, and choose the logarithmic utility function. Note that in this case there is full depreciation. Perform numerical value function iteration in MatLab. Plot value functions obtained in different stages. Does the procedure converge?

(2) Plot on the same graph value function obtained through guess–and–verify procedure (it was done during lecture, but you can replicate calculations by yourself). Does the value function obtained in the previous step converge to its theoretical counterpart? (Hint: To plot an analytical function simple pass the k–grid as argument of function and plot the result.)

(3) Plot optimal policy function obtained in matlab and by guess and verify in the separate graph. Do they coincide?

(4) Now set $\delta = 0.1$. Obtain numerically the value function and policy function as in the previous case and compare them. Comment on the differences.

(5) Now set α to 0.9. Obtain numerically the value function and policy function as in the previous case and compare them. Comment on the differences.

(6) Take the parameters as in part (4) and change the utility function to CRRA utility function:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\gamma} - 1}{1-\gamma}\right), \quad \textit{where } \gamma = 10.$$

Obtain value function and policy function and compare with resutls from part (4). What is the intuition behind the differences?

Solution: See separate Matlab file.