Macro Topics: Introduction to Matlab
Fall 2016
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Homework #2

Suggested Solutions

Problem 1. *Consider the following habit persistence problem:*

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t, c_{t-1})$$

subject to the following constraints:

$$k_{t+1} + c_t \leq Ak_t^{\alpha}, \quad A > 0, \ \ \alpha \in (0,1),$$

$$c_t > 0, \ \ k_t > 0, \ \ k_0, c_{-1} \text{ given.}$$

(1) Identify state and control variables and write down the Bellman Equation for this problem.

(2) Derive FOC, ET and EE (note that this time the habit is not additively separable and hence the final result will differ from the one we obtained during the session). When deriving the ET with respect to the capital, do not simply apply the envelope theorem, but rather derive it explicitly (as we have done during the lecture).

(3) Derive the steady state vlaues of consumption and capital. Do they differ from the steady state values we obtained during the session?

Solution:

(1) The state and control variables are:

- states: k_t, c_{t-1};
- controls: c_t, k_{t+1}.

The Bellman Equation has the following form:

$$V(k, c_{-1}) = \max_{\tilde{k}, c}\{u(c, c_{-1}) + \beta V(\tilde{k}, c)\} \ \text{ s.t } \ c = Ak^{\alpha} - \tilde{k}.$$

Or after substitution:

$$V(k, c_{-1}) = \max_{\tilde{k}}\{u(Ak^{\alpha} - \tilde{k}, c_{-1}) + \beta V(\tilde{k}, c)\}.$$

(2) We proceed as usually:

$$FOC: \quad -u_1(c, c_{-1}) + \beta[V_1(\tilde{k}, c) - V_2(\tilde{k}, c)] = 0,$$

$$\text{ET w.r.t. } k: \quad \frac{\partial V(k, c_{-1})}{\partial k} = u_1(c, c_{-1})\left[\frac{\partial c(k, \tilde{k}(k))}{\partial k} + \frac{\partial c(k, \tilde{k}(k))}{\partial \tilde{k}}\frac{\partial \tilde{k}}{\partial k}\right] +$$

$$+\beta\left\{\frac{\partial V(\tilde{k}(k),c)}{\partial \tilde{k}}\frac{\partial \tilde{k}}{\partial k}+\frac{\partial V(\tilde{k}(k),c)}{\partial c}\left[\frac{\partial c(k,\tilde{k}(k))}{\partial k}+\frac{\partial c(k,\tilde{k}(k))}{\partial \tilde{k}}\frac{\partial \tilde{k}}{\partial k}\right]\right\}.$$

The terms that include ${}^{\partial\tilde{k}}/_{\partial k}$ can be collected into FOC and hence yield zero. The envelope condition then simplifies into:

$$\frac{\partial V(k,c_{-1})}{\partial k}=u_1(c,c_{-1})\alpha Ak^{\alpha-1}+\beta\frac{\partial V(\tilde{k},c)}{\partial c}\alpha Ak^{\alpha-1}.$$

The second envelope theorem is easy, since there are no interactions between c_{-1} and the choice and state variables:

$$\text{ET w.r.t. } c_{-1}: \quad \frac{\partial V(k,c_{-1})}{\partial c_{-1}}=u_2(c,c_{-1}).$$

Since we have partial of future value function w.r.t. c in the first envelope condition, we have to substitute it there (after forwarding it one period) before substituting that envelope condition into FOC:

$$\frac{\partial V(k,c_{-1})}{\partial k}=u_1(c,c_{-1})\alpha Ak^{\alpha-1}+\beta u_2(\tilde{c},c)\alpha Ak^{\alpha-1}.$$

Substituting both envelope conditions (in correct timing) into the FOC yields:

$$u_1(c,c_{-1})=\beta\left\{\alpha A\tilde{k}^{\alpha-1}[u_1(\tilde{c},c)+\beta u_2(\tilde{\tilde{c}},\tilde{c})]-u_2(\tilde{c},c)\right\}.$$

Now rearranging we obtain Euler Equation:

$$u_1(c,c_{-1})+\beta u_2(\tilde{c},c)=\alpha\beta A\tilde{k}^{\alpha-1}[u_1(\tilde{c},c)+\beta u_2(\tilde{\tilde{c}},\tilde{c})].$$

Crucially, in contrast to the additive separable case from the exercise session, now we cannot cancel the corresponding terms.

(3) The transition law does not change from the additively separable case, so only difference is in the EE. This becomes in steady state:

$$u_1(\bar{c},\bar{c})+\beta u_2(\bar{c},\bar{c})=\alpha\beta A\bar{k}^{\alpha-1}[u_1(\bar{c},\bar{c})+\beta u_2(\bar{c},\bar{c})],$$

$$1=\alpha\beta A\bar{k}^{\alpha-1} \;\Rightarrow\; \bar{k}=(A\alpha\beta)^{1/(1-\alpha)}.$$

For consumption we use the transiton law:

$$\bar{c}=A\bar{k}^{\alpha}-\bar{k}=(A\alpha\beta)^{1/(1-\alpha)}\left(\frac{1}{\alpha\beta}-1\right).$$

This result proves that internal (as oposed to external) habit persistence does not affect the steady state, but only dynamics, since the agent internalizes the effects on the habit persistance.

Problem 2. *Consider the following modification of Robinson Crusoe problem:*

$$\max_{\{c_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t\left(\frac{c_t^{1-\gamma}-1}{1-\gamma}\right) \quad s.t.\; c_t=Ak_t^{\alpha}+(1-\delta)k_t-k_{t+1}.$$

(1) Identify state and control variables and write down the Bellman Equation for this problem.

(2) Derive FOC, ET and EE.

(3) Derive the steady state vlaues of consumption and capital.

Solution:

(1) The state and control variables are:

- states: k_t;
- controls: c_t, k_{t+1}.

The Bellman Equation has the following form:

$$V(k) = \max_{\tilde{k},c} \left\{ \frac{c^{1-\gamma} - 1}{1-\gamma} + \beta V(\tilde{k}) \right\} \text{ s.t } c = Ak^\alpha + (1-\delta)k - \tilde{k}.$$

Or after substitution:

$$V(k) = \max_{\tilde{k}} \left\{ \frac{(Ak^\alpha + (1-\delta)k - \tilde{k})^{1-\gamma} - 1}{1-\gamma} + \beta V(\tilde{k}) \right\}.$$

(2) We proceed as usually:

$$FOC: \quad -(Ak^\alpha + (1-\delta)k - \tilde{k})^{-\gamma} + \beta V'(\tilde{k}) = 0,$$

$$ET: \quad V'(k) = (Ak^\alpha + (1-\delta)k - \tilde{k})^{-\gamma}(A\alpha k^{\alpha-1} + 1 - \delta).$$

Shift ET condition one period ahead:

$$V'(\tilde{k}) = (A\tilde{k}^\alpha + (1-\delta)\tilde{k} - \tilde{\tilde{k}})^{-\gamma}(A\alpha\tilde{k}^{\alpha-1} + 1 - \delta).$$

Substituting the shifted envelope condition into the FOC yields Euler Equation:

$$-(Ak^\alpha + (1-\delta)k - \tilde{k})^{-\gamma} + \beta(A\tilde{k}^\alpha + (1-\delta)\tilde{k} - \tilde{\tilde{k}})^{-\gamma}(A\alpha\tilde{k}^{\alpha-1} + 1 - \delta) = 0.$$

(3) In steady state we have $k = \tilde{k} = \tilde{\tilde{k}} = \bar{k}$ and $c = \bar{c}$. EE becomes in steady state:

$$-(A\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{k})^{-\gamma} + \beta(A\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{k})^{-\gamma}(A\alpha\bar{k}^{\alpha-1} + 1 - \delta) = 0,$$

$$\frac{1}{\beta} = A\alpha\bar{k}^{\alpha-1} + 1 - \delta \quad \Rightarrow \quad \bar{k} = \left(\frac{A\alpha}{1/\beta - 1 + \delta} \right)^{1/(1-\alpha)}.$$

For consumption we use the transiton law:

$$\bar{c} = A\bar{k}^\alpha + (1-\delta)\bar{k} - \bar{k} = \left(\frac{A\alpha}{1/\beta - 1 + \delta} \right)^{1/(1-\alpha)} \left(\frac{1/\beta - 1 + \delta}{\alpha} - \delta \right).$$