Macro Topics: Introduction to Matlab
Fall 2016
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Homework #1

Suggested Solutions

Problem 1. *Consider Inventory Control problem (see Bertsekas pp.3–4 for details), where*

- x_k *— stock of a good available at the beginning of* k*–th period*
- u_k *— stock ordered at the beginning of* k*–th peroiod*
- w_k *— demand during the* k*–th period (possibly random variable)*

The stock evolves according to the discrete time equation:

$$x_{k+1} = x_k + u_k - w_k,$$

where negative stock corresponds to backlogged demand.

Firm is functioning for N *periods. The purchasing costs in every period are equal* cu_k *and penalty for non–zero stock is equal* $(x_k + u_k - w_k)^2$*, thus, costs incurred in period* k *are sum of those two. Suppose that there are no costs at the last period, i.e.* $g(x_N) = 0$*. Suppose also that* $w_k = 10$ *for every* k*.*

(1) Solve the problem for $N = 2$*, i.e. find optimal policies in both periods and cost–to–go functions.*

(2) Find optimal policies and cost–to–go functions for arbitrary N*.*

(3) How will your answer change if w_k *is a random variable with mean 10 and variance 4 (*$\mathbb{E}[w_k] = 10$, $\mathbb{E}[w_k^2] - (\mathbb{E}[w_k])^2 = 4$*)?*

(4) How does the stock after delivery (i.e. $x_k + u_k$*) depend on the current period number?*

Solution:

(1) Since there is no cost in the last period, cost–to–go function in the end of the second period $J_2(x_2) = 0$. Using the principle of optimality, consider the problem, where firm solves the following problem in the beginning of the period one:

$$J_1(x_1) = \min_{u_1}\{g_1(x_1, u_1) + J_2(x_2)\} = \min_{u_1}\{g_1(x_1, u_1)\} = \min_{u_1}\{cu_1 + (x_1 + u_1 - 10)^2\}.$$

Taking F.O.C. with respect to u_1 (assuming u_1 can be any real number), we get:

$$c + 2(x_1 + u_1 - 10) = 0,$$

and after some rearranging

$$u_1 = 10 - x_1 - \frac{c}{2}.$$

Notice that u_1 remains positive uif $x_1 \leq 10 - {}^{c}\!/_{2}$. Plugging this policy function back into the firm's problem, we obtain:

$$J_1(x_1) = c(10 - x_1 - \frac{c}{2}) + (x_1 + 10 - x_1 - \frac{c}{2} - 10)^2 = c(10 - x_1) - \frac{c^2}{4}.$$

Now we have cost–to–go function for the period one and we can solve the firm's problem in the period zero:

$$J_0(x_0) = \min_{u_0}\{g_0(x_0, u_0) + J_1(x_1)\} = \min_{u_0}\{cu_0 + (x_0 + u_0 - 10)^2 + J_1(x_0 + u_0 - 10)\} =$$

$$= \min_{u_0}\{cu_0 + (x_0 + u_0 - 10)^2 + c(10 - (x_0 + u_0 - 10)) - \frac{c^2}{4}\} =$$

$$= \min_{u_0}\{(x_0 + u_0 - 10)^2 + c(20 - x_0) - \frac{c^2}{4}\}.$$

Again, take the F.O.C. and obtain:

$$2(x_0 + u_0 - 10) = 0$$

or

$$u_0 = 10 - x_0.$$

Notice that u_0 is positive if $x_0 < 10$. Plugging this result back to the cost function, we obtain cost function for the firm:

$$J_0(x_0) = c(20 - x_0) - \frac{c^2}{4}.$$

(2) First, notice that independently of a number of periods, solution for the second to the last periods remains the same, i.e. $J_{N-1}(x_{N-1}) = c(10 - x_{N-1}) - {}^{c^2}\!/_{4}$. Observing some similarities from the item (1), we can hypothesize that

$$J_k(x_k) = c(10(N - k) - x_k) - \frac{c^2}{4},$$

which is obviously true for $k = N - 1$. Using induction principle, assume that it is also true for some $k+1$. What cost function for the period k follows from this assumption?

$$J_k(x_k) = \min_{u_k}\{g_k(x_k, u_k) + J_{k+1}(x_{k+1})\} = \min_{u_k}\{cu_k + (x_k + u_k - 10)^2 + J_{k+1}(x_k + u_k - 10)\} =$$

$$= \min_{u_k}\{cu_k + (x_k + u_k - 10)^2 + c(10(N - (k + 1)) - (x_k + u_k - 10)) - \frac{c^2}{4}\} =$$

$$= \min_{u_k}\{(x_k + u_k - 10)^2 + c(10(N - k) - x_k) - \frac{c^2}{4}\}.$$

The F.O.C. immediately delivers:

$$2(x_k + u_k - 10) = 0$$

or

$$u_k = 10 - x_k.$$

And plugging it back we get:

$$J_k(x_k) = c(10(N-k) - x_k) - \frac{c^2}{4},$$

proving our hypothesis. In other words, in every period except the second last, firm exactly meets demand. It is easy to understand since demand is backlogging and all the left demand will occur in the next period.

(3) With quadratic objective function and linear law of motion for states, only mean of the random variable matters for policy function. Hence, since $\mathbb{E}[w] = 10$, policy functions remain unchanged. However, it is worth understanding how cost functions are affected by random demand. Suppose, we have some cost–to–go function $J_{k+1}(x_{k+1})$. Then in period k firm solves the following problem:

$$J_k(x_k) = \min_{u_k}\{\mathbb{E}_{w_k}[cu_k + (x_k + u_k - w_k)^2 + J(x_{k+1})]\} =$$

$$= \min_{u_k}\{\mathbb{E}_{w_k}[cu_k + (x_k + u_k)^2 - 2w_k(x_k + u_k) + w_k^2 + J(x_{k+1})]\} =$$

$$= \min_{u_k}\{cu_k + (x_k + u_k)^2 - 2\mathbb{E}[w_k](x_k + u_k) + \mathbb{E}[w_k^2] + \mathbb{E}_{w_k}[J(x_k + u_k - w_k)]\}.$$

Suppose for a moment that cost–to–go functions are linear in demand and, hence, $\mathbb{E}_{w_k}[J(x_k + u_k - w_k)] = J(x_k + u_k - \mathbb{E}[w_k])$. In this case we can rewrite:

$$J_k(x_k) = \min_{u_k}\{cu_k + (x_k + u_k)^2 - 2\mathbb{E}[w_k](x_k + u_k) + (\mathbb{E}[w_k])^2 - (\mathbb{E}[w_k])^2 +$$

$$+ \mathbb{E}[w_k^2] + \mathbb{E}_{w_k}[J(x_k + u_k - w_k)]\} =$$

$$= \min_{u_k}\{cu_k + (x_k + u_k - \mathbb{E}[w_k])^2 + J(x_k + u_k - \mathbb{E}[w_k])\} + \mathbb{E}[w_k^2] - (\mathbb{E}[w_k])^2.$$

As we see, we only add variance of demand to the costs in every period. It is now straightforward to show by induction argument that

$$J_k(x_k) = c(10(N-k) - x_k) - \frac{c^2}{4} + (N-k)\mathrm{Var}(w_k).$$

(4) As we have shown already, firm tries to meet demand in every period but the second last. Hence, $\forall k < N-1$ $x_k + u_k = 10$ and as follows from item (1) $u_{N-1} + x_{N-1} = 10 - c/2$.