Macro Topics: Introduction to Matlab
Fall 2016
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Exercise Session #3

Suggested Solutions

Problem 1. *Consider the following model of economic growth. The world is deterministic. Time is discrete. Representative dynasty maximizes lifetime welfare given by*

$$\sum_{t=0}^{\infty} \beta^t (\log c_t - \eta l_t), \quad h_0 \text{ given,}$$

where c_t is consumption, l_t is labour effort, $\eta > 0$, and discount factor $\beta \in (0,1)$. The household allocates it's resources (disposable income) $\hat{y}_t$ between consumption and investment into human capital, e_t. Disposable income $\hat{y}_t$ is given as

$$\hat{y}_t = (1-\tau) y_t, \tag{1}$$

where $\tau \in [0,1]$ is a proportional tax rate collected by the government.

Pre–tax income of the household is given by

$$y_t = h_t^{\lambda} l_t^{1-\lambda}, \tag{2}$$

where $0 < \lambda < 1$. Human capital of the household accumulates according to

$$h_{t+1} = [(1+\alpha)e_t]^{\gamma}, \tag{3}$$

where constant α is the proportional subsidy paid by the government to support human capital investment, and $0 < \gamma < 1$.

As is seen from the description of the model, the only form of "savings" is possible in form of spending resources on producing human capital h_t. This could be justified if we assume that the output y_t consists of a perishable good. Current human capital h_t does not influence future h_{t+1}, therefore, "human capital" can be thought of as a person's skills that fully depreciate within a period.

Household takes parameters of the tax/subsidy system τ and α as given.

1. *Clearly identify state and control variable(s). Set up the Bellman equation for the problem (that is write the problem in the recursive form).*
2. *Derive First order condition(s) and Envelope Theorem condition(s).*
3. *Use your ET and FOC to derive the share of educational expenditures in the disposable income, or "savings rate", which would prevail in the steady state:*

$$s^* = \frac{e^*}{\hat{y}^*}.$$

4. *Derive steady state values of labour effort, consumption, and human capital.*

Solution:

1. We start by eliminating as much of the variables as possible. First of all, since household allocates it's resources (disposable income) $\hat{y}_t$ between consumption and investment into human capital, e_t, the following equation holds:

$$\hat{y}_t = c_t + e_t \tag{4}$$

   Plugging (4) into the LHS of equation (1) and substituting y_t from the equation (2) into the RHS of (1), we get:

$$c_t + e_t = (1-\tau)h_t^{\lambda} l_t^{1-\lambda}.$$

   Moving the equation (3) one period back and substituting into the equation above, we get:

$$c_t + e_t = (1-\tau)(1+\alpha)^{\gamma\lambda} e_{t-1}^{\gamma\lambda} l_t^{1-\lambda}.$$

   We can rewrite the last equation in the following way:

$$c_t = (1-\tau)(1+\alpha)^{\gamma\lambda} e_{t-1}^{\gamma\lambda} l_t^{1-\lambda} - e_t. \tag{5}$$

   We can identify the remaining state and control variables:
   - **states:** e_{t-1},
   - **controls:** c_t, l_t, e_t.

   We are now ready to set up the Bellman equation for the problem:

$$V(e_{t-1}) = \max_{e_t, l_t} \{\log c_t - \eta l_t + \beta V(e_t)\}$$

   (to avoid messy expressions we will keep c_t, even though we will not treat it as a control variable anymore).

2. First we take the FOCs (since we have two control variables we will also have two FOCs):

$$[e_t]: \quad -\frac{1}{c_t} + \beta V'(e_t) = 0, \tag{6}$$

$$[l_t]: \quad \frac{1}{c_t}(1-\lambda)(1-\tau)(1+\alpha)^{\gamma\lambda} e_{t-1}^{\gamma\lambda} l_t^{-\lambda} - \eta = 0. \tag{7}$$

   The ET condition:

$$[e_{t-1}]: \quad V'(e_{t-1}) = \frac{1}{c_t}\gamma\lambda(1-\tau)(1+\alpha)^{\gamma\lambda} e_{t-1}^{\gamma\lambda-1} l_t^{1-\lambda}. \tag{8}$$

3. From now on assume that we are in the steady state. Denote by

$$l = l_t = l_{t-1},$$

$$c = c_t = c_{t-1},$$

$$e = e_t = e_{t-1}$$

   the steady–state values of l, c, and e respectively.

We plug in the equation (8) into (6) to get:

$$\beta\gamma\lambda\frac{1}{c}(1-\tau)(1+\alpha)^{\gamma\lambda}e^{\gamma\lambda-1}l^{1-\lambda}=\frac{1}{c}.$$

Using (5) (in the steady state), this simplifies to

$$\beta\gamma\lambda\frac{1}{e}(c+e)=1,$$

or

$$s=\beta\gamma\lambda.$$

4. Rewriting equation (7)

$$\frac{c+e}{cl}(1-\lambda)=\eta,$$

and using the derived steady–state value for s, we get

$$\frac{e}{cl}(1-\lambda)=\eta\beta\gamma\lambda \quad \Rightarrow \quad l=\frac{\left({}^{1}\!/_{\beta\gamma\lambda}-1\right)(1-\lambda)}{\eta\beta\lambda\gamma}.$$

We now can use the equation (5) and the equation for s to solve for the steady–state values of e and c:

$$e=\left(\left(\frac{\eta\beta\lambda\gamma}{\left({}^{1}\!/_{\beta\gamma\lambda}-1\right)(1-\lambda)}\right)^{1-\lambda}\frac{1}{\beta\gamma\lambda(1-\tau)(1+\alpha)^{\gamma\lambda}}\right)^{1/(\gamma\lambda-1)},$$

$$c=\left(\frac{1}{\beta\gamma\lambda}-1\right)\left(\left(\frac{\eta\beta\lambda\gamma}{\left({}^{1}\!/_{\beta\gamma\lambda}-1\right)(1-\lambda)}\right)^{1-\lambda}\frac{1}{\beta\gamma\lambda(1-\tau)(1+\alpha)^{\gamma\lambda}}\right)^{1/(\gamma\lambda-1)}.$$

Problem 2. *Consider the problem of the optimal harvesting of the renewable resource. Social planner seeks to maximize the lifetime utility of the representative consumer*

$$\sum_{t=0}^{\infty}u(C_t,P_t)$$

subject to

$$Z_{t+1}=Z_t+Z_t(K-Z_t)-S_t,$$
$$Z_t\geq 0,\quad S_t\geq 0,$$

where Z is current stock of renewable resource, S is the harvesting rate. Utility function of representative consumer depends on consumption C and consumer gets disutility from environmental damage P. Consumption good is non–storable and is produced using production function:

$$Y_t=S_t^{\alpha},\quad \alpha<1.$$

Environmental damage is proportional to harvesting rate, that is

$$P_t=\rho S_t.$$

1. *Clearly identify state and control variables. Rewrite the problem in the following form:*

$$\max_{x_t} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}),$$

where x_t is the state variable.

2. *Set up the Bellman equation for the problem (that is write the problem in the recursive form).*
3. *Derive First order conditions and Envelope Theorem conditions.*
4. *Using the equations from above derive Euler equation and calculate steady–state values of Z and S.*

Solution:

1. We will start by putting all the implicitly stated conditions into the equations. Since good is non–storable, we have $C_t = Y_t = S_t^{\alpha}$. Now, we have:
   - **states:** Z_t,
   - **controls:** C_t, S_t, Z_{t+1}, P_t, Y_t.

   Using the equations for C_t and P_t we can eliminate S_t and Y_t to get

$$C_t = (Z_t - Z_{t+1} + Z_t(K - Z_t))^{\alpha},$$

$$P_t = \rho(Z_t - Z_{t+1} + Z_t(K - Z_t)).$$

   Choosing Z_t as our state, we can now rewrite our problem in the form

$$\max_{x_t} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}),$$

   by plugging equations for C_t and P_t into the objective. We get:

$$\sum_{t=0}^{\infty} \beta^t u(C_t(Z_t, Z_{t+1}), P_t(Z_t, Z_{t+1})).$$

2. We can now rewrite the problem in the recursive form. Denote $Z_t = Z$ and $Z_{t+1} = \tilde{Z}$. Then the Bellman equation is

$$V(Z) = \max_{\tilde{Z}} \left\{ u(C(Z, \tilde{Z}), P(Z, \tilde{Z})) + \beta V(\tilde{Z}) \right\}.$$

3. Again, for simplicity denote $u_i(x_1, ..., x_n) = {}^{\partial u(x_1,...,x_n)}/_{\partial x_i}$, $i = 1...n$ for some function u.

   FOC (derivative with respect to $\tilde{Z}$) for Bellman equation is

$$u_1(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot C_2(Z, \tilde{Z}) + u_2(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot P_2(Z, \tilde{Z}) + \beta V'(\tilde{Z}) = 0.$$

   ET condition (derivative with respect to Z) is given by the equation

$$V'(Z) = u_1(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot C_1(Z, \tilde{Z}) + u_2(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot P_1(Z, \tilde{Z}).$$

4. As usually, we shift the above equation one period ahead to get

$$V'(\tilde{Z}) = u_1(C(\tilde{Z},\tilde{\tilde{Z}}), P(\tilde{Z},\tilde{\tilde{Z}})) \cdot C_1(\tilde{Z},\tilde{\tilde{Z}}) + u_2(C(\tilde{Z},\tilde{\tilde{Z}}), P(\tilde{Z},\tilde{\tilde{Z}})) \cdot P_1(\tilde{Z},\tilde{\tilde{Z}}).$$

Plugging this into the FOC we can get Euler Equation

$$u_1(C(Z,\tilde{Z}), P(Z,\tilde{Z})) \cdot C_2(Z,\tilde{Z}) + u_2(C(Z,\tilde{Z}), P(Z,\tilde{Z})) \cdot P_2(Z,\tilde{Z}) +$$
$$+ \beta \left(u_1(C(\tilde{Z},\tilde{\tilde{Z}}), P(\tilde{Z},\tilde{\tilde{Z}})) \cdot C_1(\tilde{Z},\tilde{\tilde{Z}}) + u_2(C(\tilde{Z},\tilde{\tilde{Z}}), P(\tilde{Z},\tilde{\tilde{Z}})) \cdot P_1(\tilde{Z},\tilde{\tilde{Z}}) \right) = 0.$$

Since we know that

$$C = \left(Z - \tilde{Z} + Z(K - Z)\right)^{\alpha},$$
$$P = \rho\left(Z - \tilde{Z} + Z(K - Z)\right),$$

we can calculate all the derivatives in EE:

$$C_1(Z,\tilde{Z}) = \alpha(1 + K - 2Z)\left(Z - \tilde{Z} + Z(K - Z)\right)^{\alpha-1},$$
$$C_2(Z,\tilde{Z}) = -\alpha\left(Z - \tilde{Z} + Z(K - Z)\right)^{\alpha-1},$$
$$P_1(Z,\tilde{Z}) = \rho(1 + K - 2Z),$$
$$P_2(Z,\tilde{Z}) = -\rho,$$

and we can plug these into the equation. In steady state $Z = \tilde{Z} = \tilde{\tilde{Z}} = Z^*$. We have the following equations to determine the steady–state values:

$$Z^* = Z^* + Z^*(K - Z^*) - S^* \quad \Rightarrow \quad S^* = Z^*(K - Z^*),$$
$$\left(\beta(1 + K - 2Z^*) - 1\right)\left(\alpha(Z^*(K - Z^*))^{\alpha-1}u_1(Z^*) + \rho u_2(Z^*)\right),$$

where the last equation is the Euler Equation evaluated at steady–state values of Z^*.

Problem 3. ***(Failure of Guess–and–Verify Method)*** *Consider the household that seeks to maximize his lifetime utility*

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1 - \sigma},$$

subject to the following constraint:

$$k_{t+1} + c_t = Ak_t^{\alpha} + (1 - \delta)k_t, \quad 0 < \delta < 1.$$

Show that guessing that policy function is

- *constant, that is $V(k_t) = C$,*
- *linear in state (excluding constant), that is $V(k_t) = Ck_t$*

does not work.

Solution: First assume that our guess is $V(k_t) = C$, where C is some constant. Then, Bellman equation is

$$V(k_t) = \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{1-\sigma} - 1}{1-\sigma} + \beta V(k_{t+1}) \right\} =$$

$$= \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{1-\sigma} - 1}{1-\sigma} + \beta C \right\}.$$

Taking the FOC (the derivative with respect to k_{t+1}) we get:

$$-(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{-\sigma} = 0,$$

which has unique solution when $k_{t+1} = Ak_t^\alpha + (1-\delta)k_t \Rightarrow c_t = 0$. However, this is not optimal, and thus we conclude that our guess was wrong.

Now assume that our guess is that $V(k_t) = Ck_t$. Again, Bellman equation is

$$V(k_t) = \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{1-\sigma} - 1}{1-\sigma} + \beta V(k_{t+1}) \right\} =$$

$$= \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{1-\sigma} - 1}{1-\sigma} + \beta C k_{t+1} \right\}.$$

Taking the FOC (the derivative with respect to k_{t+1}) we get:

$$-(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{-\sigma} + \beta C = 0 \quad \Rightarrow \quad k_{t+1} = Ak_t^\alpha + (1-\delta)k_t - (\beta C)^{-1/\sigma}.$$

Plugging this back into Bellman equation, and equating terms next to k_t^α we get that

$$\beta A C = 0,$$

which implies $C = 0$. Thus, the value function is constant and equal to zero. We obtained a contradiction. Therefore, our guess was wrong.

Problem 4. ***(Matlab)*** *Compute the value of the period reward function, which in our case is only the utility function of the following form:*

$$\log c_t + \rho \log c_{t-1}.$$

In maximizing his utility, the consumer is constrained by the following budget constraint:

$$Ak_t^\alpha + (1-\delta)k_t - k_{t+1} = c_t.$$

Solution: See separate Matlab file.