

## Exercise Session #3

### Suggested Solutions

**Problem 1.** Consider the following model of economic growth. The world is deterministic. Time is discrete. Representative dynasty maximizes lifetime welfare given by

$$\sum_{t=0}^{\infty} \beta^t (\log c_t - \eta l_t), \quad h_0 \text{ given,}$$

where  $c_t$  is consumption,  $l_t$  is labour effort,  $\eta > 0$ , and discount factor  $\beta \in (0, 1)$ . The household allocates its resources (disposable income)  $\hat{y}_t$  between consumption and investment into human capital,  $e_t$ . Disposable income  $\hat{y}_t$  is given as

$$\hat{y}_t = (1 - \tau)y_t, \tag{1}$$

where  $\tau \in [0, 1]$  is a proportional tax rate collected by the government.

Pre-tax income of the household is given by

$$y_t = h_t^\lambda l_t^{1-\lambda}, \tag{2}$$

where  $0 < \lambda < 1$ . Human capital of the household accumulates according to

$$h_{t+1} = [(1 + \alpha)e_t]^\gamma, \tag{3}$$

where constant  $\alpha$  is the proportional subsidy paid by the government to support human capital investment, and  $0 < \gamma < 1$ .

As is seen from the description of the model, the only form of “savings” is possible in form of spending resources on producing human capital  $h_t$ . This could be justified if we assume that the output  $y_t$  consists of a perishable good. Current human capital  $h_t$  does not influence future  $h_{t+1}$ , therefore, “human capital” can be thought of as a person’s skills that fully depreciate within a period.

Household takes parameters of the tax/subsidy system  $\tau$  and  $\alpha$  as given.

1. Clearly identify state and control variable(s). Set up the Bellman equation for the problem (that is write the problem in the recursive form).
2. Derive First order condition(s) and Envelope Theorem condition(s).
3. Use your ET and FOC to derive the share of educational expenditures in the disposable income, or “savings rate”, which would prevail in the steady state:

$$s^* = \frac{e^*}{\hat{y}^*}.$$

4. Derive steady state values of labour effort, consumption, and human capital.

**Solution:**

1. We start by eliminating as much of the variables as possible. First of all, since household allocates its resources (disposable income)  $\hat{y}_t$  between consumption and investment into human capital,  $e_t$ , the following equation holds:

$$\hat{y}_t = c_t + e_t \quad (4)$$

Plugging (4) into the LHS of equation (1) and substituting  $y_t$  from the equation (2) into the RHS of (1), we get:

$$c_t + e_t = (1 - \tau)h_t^\lambda l_t^{1-\lambda}.$$

Moving the equation (3) one period back and substituting into the equation above, we get:

$$c_t + e_t = (1 - \tau)(1 + \alpha)^\gamma e_{t-1}^{\gamma\lambda} l_t^{1-\lambda}.$$

We can rewrite the last equation in the following way:

$$c_t = (1 - \tau)(1 + \alpha)^\gamma e_{t-1}^{\gamma\lambda} l_t^{1-\lambda} - e_t. \quad (5)$$

We can identify the remaining state and control variables:

- **states:**  $e_{t-1}$ ,
- **controls:**  $c_t, l_t, e_t$ .

We are now ready to set up the Bellman equation for the problem:

$$V(e_{t-1}) = \max_{e_t, l_t} \{ \log c_t - \eta l_t + \beta V(e_t) \}$$

(to avoid messy expressions we will keep  $c_t$ , even though we will not treat it as a control variable anymore).

2. First we take the FOCs (since we have two control variables we will also have two FOCs):

$$[e_t]: \quad -\frac{1}{c_t} + \beta V'(e_t) = 0, \quad (6)$$

$$[l_t]: \quad \frac{1}{c_t}(1 - \lambda)(1 - \tau)(1 + \alpha)^\gamma e_{t-1}^{\gamma\lambda} l_t^{-\lambda} - \eta = 0. \quad (7)$$

The ET condition:

$$[e_{t-1}]: \quad V'(e_{t-1}) = \frac{1}{c_t} \gamma \lambda (1 - \tau) (1 + \alpha)^\gamma e_{t-1}^{\gamma\lambda - 1} l_t^{1-\lambda}. \quad (8)$$

3. From now on assume that we are in the steady state. Denote by

$$l = l_t = l_{t-1},$$

$$c = c_t = c_{t-1},$$

$$e = e_t = e_{t-1}$$

the steady-state values of  $l$ ,  $c$ , and  $e$  respectively.

We plug in the equation (8) into (6) to get:

$$\beta\gamma\lambda\frac{1}{c}(1-\tau)(1+\alpha)^{\gamma\lambda}e^{\gamma\lambda-1}l^{1-\lambda} = \frac{1}{c}.$$

Using (5) (in the steady state), this simplifies to

$$\beta\gamma\lambda\frac{1}{e}(c+e) = 1,$$

or

$$s = \beta\gamma\lambda.$$

4. Rewriting equation (7)

$$\frac{c+e}{cl}(1-\lambda) = \eta,$$

and using the derived steady-state value for  $s$ , we get

$$\frac{e}{cl}(1-\lambda) = \eta\beta\gamma\lambda \Rightarrow l = \frac{(1/\beta\gamma\lambda - 1)(1-\lambda)}{\eta\beta\gamma\lambda}.$$

We now can use the equation (5) and the equation for  $s$  to solve for the steady-state values of  $e$  and  $c$ :

$$e = \left( \left( \frac{\eta\beta\gamma\lambda}{(1/\beta\gamma\lambda - 1)(1-\lambda)} \right)^{1-\lambda} \frac{1}{\beta\gamma\lambda(1-\tau)(1+\alpha)^{\gamma\lambda}} \right)^{1/(\gamma\lambda-1)},$$

$$c = \left( \frac{1}{\beta\gamma\lambda} - 1 \right) \left( \left( \frac{\eta\beta\gamma\lambda}{(1/\beta\gamma\lambda - 1)(1-\lambda)} \right)^{1-\lambda} \frac{1}{\beta\gamma\lambda(1-\tau)(1+\alpha)^{\gamma\lambda}} \right)^{1/(\gamma\lambda-1)}.$$

**Problem 2.** Consider the problem of the optimal harvesting of the renewable resource. Social planner seeks to maximize the lifetime utility of the representative consumer

$$\sum_{t=0}^{\infty} u(C_t, P_t)$$

subject to

$$Z_{t+1} = Z_t + Z_t(K - Z_t) - S_t,$$

$$Z_t \geq 0, \quad S_t \geq 0,$$

where  $Z$  is current stock of renewable resource,  $S$  is the harvesting rate. Utility function of representative consumer depends on consumption  $C$  and consumer gets disutility from environmental damage  $P$ . Consumption good is non-storable and is produced using production function:

$$Y_t = S_t^\alpha, \quad \alpha < 1.$$

Environmental damage is proportional to harvesting rate, that is

$$P_t = \rho S_t.$$

1. Clearly identify state and control variables. Rewrite the problem in the following form:

$$\max_{x_t} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}),$$

where  $x_t$  is the state variable.

2. Set up the Bellman equation for the problem (that is write the problem in the recursive form).
3. Derive First order conditions and Envelope Theorem conditions.
4. Using the equations from above derive Euler equation and calculate steady-state values of  $Z$  and  $S$ .

**Solution:**

1. We will start by putting all the implicitly stated conditions into the equations. Since good is non-storable, we have  $C_t = Y_t = S_t^\alpha$ . Now, we have:

- **states:**  $Z_t$ ,
- **controls:**  $C_t, S_t, Z_{t+1}, P_t, Y_t$ .

Using the equations for  $C_t$  and  $P_t$  we can eliminate  $S_t$  and  $Y_t$  to get

$$C_t = (Z_t - Z_{t+1} + Z_t(K - Z_t))^\alpha,$$

$$P_t = \rho(Z_t - Z_{t+1} + Z_t(K - Z_t)).$$

Choosing  $Z_t$  as our state, we can now rewrite our problem in the form

$$\max_{x_t} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}),$$

by plugging equations for  $C_t$  and  $P_t$  into the objective. We get:

$$\sum_{t=0}^{\infty} \beta^t u(C_t(Z_t, Z_{t+1}), P_t(Z_t, Z_{t+1})).$$

2. We can now rewrite the problem in the recursive form. Denote  $Z_t = Z$  and  $Z_{t+1} = \tilde{Z}$ . Then the Bellman equation is

$$V(Z) = \max_{\tilde{Z}} \left\{ u(C(Z, \tilde{Z}), P(Z, \tilde{Z})) + \beta V(\tilde{Z}) \right\}.$$

3. Again, for simplicity denote  $u_i(x_1, \dots, x_n) = \partial u(x_1, \dots, x_n) / \partial x_i$ ,  $i = 1 \dots n$  for some function  $u$ .

FOC (derivative with respect to  $\tilde{Z}$ ) for Bellman equation is

$$u_1(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot C_2(Z, \tilde{Z}) + u_2(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot P_2(Z, \tilde{Z}) + \beta V'(\tilde{Z}) = 0.$$

ET condition (derivative with respect to  $Z$ ) is given by the equation

$$V'(Z) = u_1(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot C_1(Z, \tilde{Z}) + u_2(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot P_1(Z, \tilde{Z}).$$

4. As usually, we shift the above equation one period ahead to get

$$V'(\tilde{Z}) = u_1(C(\tilde{Z}, \tilde{Z}), P(\tilde{Z}, \tilde{Z})) \cdot C_1(\tilde{Z}, \tilde{Z}) + u_2(C(\tilde{Z}, \tilde{Z}), P(\tilde{Z}, \tilde{Z})) \cdot P_1(\tilde{Z}, \tilde{Z}).$$

Plugging this into the FOC we can get Euler Equation

$$u_1(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot C_2(Z, \tilde{Z}) + u_2(C(Z, \tilde{Z}), P(Z, \tilde{Z})) \cdot P_2(Z, \tilde{Z}) + \\ + \beta \left( u_1(C(\tilde{Z}, \tilde{Z}), P(\tilde{Z}, \tilde{Z})) \cdot C_1(\tilde{Z}, \tilde{Z}) + u_2(C(\tilde{Z}, \tilde{Z}), P(\tilde{Z}, \tilde{Z})) \cdot P_1(\tilde{Z}, \tilde{Z}) \right) = 0.$$

Since we know that

$$C = \left( Z - \tilde{Z} + Z(K - Z) \right)^\alpha, \\ P = \rho \left( Z - \tilde{Z} + Z(K - Z) \right),$$

we can calculate all the derivatives in EE:

$$C_1(Z, \tilde{Z}) = \alpha(1 + K - 2Z) \left( Z - \tilde{Z} + Z(K - Z) \right)^{\alpha-1}, \\ C_2(Z, \tilde{Z}) = -\alpha \left( Z - \tilde{Z} + Z(K - Z) \right)^{\alpha-1}, \\ P_1(Z, \tilde{Z}) = \rho(1 + K - 2Z), \\ P_2(Z, \tilde{Z}) = -\rho,$$

and we can plug these into the equation. In steady state  $Z = \tilde{Z} = \tilde{\tilde{Z}} = Z^*$ . We have the following equations to determine the steady-state values:

$$Z^* = Z^* + Z^*(K - Z^*) - S^* \quad \Rightarrow \quad S^* = Z^*(K - Z^*),$$

$$(\beta(1 + K - 2Z^*) - 1) (\alpha(Z^*(K - Z^*))^{\alpha-1} u_1(Z^*) + \rho u_2(Z^*)),$$

where the last equation is the Euler Equation evaluated at steady-state values of  $Z^*$ .

**Problem 3. (Failure of Guess-and-Verify Method)** Consider the household that seeks to maximize his lifetime utility

$$\max_{c_t} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma} - 1}{1-\sigma},$$

subject to the following constraint:

$$k_{t+1} + c_t = Ak_t^\alpha + (1 - \delta)k_t, \quad 0 < \delta < 1.$$

Show that guessing that policy function is

- constant, that is  $V(k_t) = C$ ,
- linear in state (excluding constant), that is  $V(k_t) = Ck_t$

does not work.

**Solution:** First assume that our guess is  $V(k_t) = C$ , where  $C$  is some constant. Then, Bellman equation is

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1 - \delta)k_t - k_{t+1})^{1-\sigma} - 1}{1 - \sigma} + \beta V(k_{t+1}) \right\} = \\ &= \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1 - \delta)k_t - k_{t+1})^{1-\sigma} - 1}{1 - \sigma} + \beta C \right\}. \end{aligned}$$

Taking the FOC (the derivative with respect to  $k_{t+1}$ ) we get:

$$-(Ak_t^\alpha + (1 - \delta)k_t - k_{t+1})^{-\sigma} = 0,$$

which has unique solution when  $k_{t+1} = Ak_t^\alpha + (1 - \delta)k_t \Rightarrow c_t = 0$ . However, this is not optimal, and thus we conclude that our guess was wrong.

Now assume that our guess is that  $V(k_t) = Ck_t$ . Again, Bellman equation is

$$\begin{aligned} V(k_t) &= \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1 - \delta)k_t - k_{t+1})^{1-\sigma} - 1}{1 - \sigma} + \beta V(k_{t+1}) \right\} = \\ &= \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1 - \delta)k_t - k_{t+1})^{1-\sigma} - 1}{1 - \sigma} + \beta Ck_{t+1} \right\}. \end{aligned}$$

Taking the FOC (the derivative with respect to  $k_{t+1}$ ) we get:

$$-(Ak_t^\alpha + (1 - \delta)k_t - k_{t+1})^{-\sigma} + \beta C = 0 \Rightarrow k_{t+1} = Ak_t^\alpha + (1 - \delta)k_t - (\beta C)^{-1/\sigma}.$$

Plugging this back into Bellman equation, and equating terms next to  $k_t^\alpha$  we get that

$$\beta AC = 0,$$

which implies  $C = 0$ . Thus, the value function is constant and equal to zero. We obtained a contradiction. Therefore, our guess was wrong.

**Problem 4. (Matlab)** Compute the value of the period reward function, which in our case is only the utility function of the following form:

$$\log c_t + \rho \log c_{t-1}.$$

In maximizing his utility, the consumer is constrained by the following budget constraint:

$$Ak_t^\alpha + (1 - \delta)k_t - k_{t+1} = c_t.$$

**Solution:** See separate Matlab file.