Macro Topics: Introduction to Matlab
Fall 2016
Instructor: Olga Bychkova
Date: 23/12/2016

Exercise Session #7

Suggested Solutions

Problem 1. ***(LS 5.1a)*** *Consider the modified version of the optimal linear regulator problem where the objective is to maximize*

$$\sum_{t=0}^{\infty} \beta^t \{x_t' R x_t + u_t' Q u_t + 2u_t' H x_t\}$$

subject to the law of motion:

$$x_{t+1} = Ax_t + Bu_t.$$

Show that the optimal policy has the form

$$u_t = -(Q + \beta B' P B)^{-1} (\beta B' P A + H) x_t,$$

where P solves the algebraic matrix Riccati equation

$$P = R + \beta A' P A - (\beta A' P B + H')(Q + \beta B' P B)^{-1} (\beta B' P A + H).$$

Solution: In order to slightly simplify our notations, let's denote

$$x = x_t, \quad \tilde{x} = x_{t+1}$$

and

$$u = u_t, \quad \tilde{u} = u_{t+1}.$$

We can set up the Bellman equation for our problem

$$V(x) = \max_u \{x' R x + u' Q u + 2u' H x + \beta V(\tilde{x})\}.$$

Making a guess that the value function has a form $V(x) = x' P x$, we can rewrite the Bellman equation in the form

$$x' P x = \max_u \{x' R x + u' Q u + 2u' H x + \beta \tilde{x}' P \tilde{x}\}.$$

Using the law of motion, we get

$$x' P x = \max_u \{x' R x + u' Q u + 2u' H x + \beta (Ax + Bu)' P (Ax + Bu)\}.$$

The FOC for u becomes

$$2Qu + 2Hx + 2\beta B' P A x + 2\beta B' P B u = 0.$$

Rearranging the last equation, we obtain

$$(Q + \beta B'PB)u = -(H + \beta B'PA)x,$$
$$u = -(Q + \beta B'PB)^{-1}(H + \beta B'PA)x,$$
$$u = -Fx, \text{ where } F = (Q + \beta B'PB)^{-1}(H + \beta B'PA).$$

Thus, we have proved the form of the optimal policy. What is left is to show that the matrix P satisfies the algebraic matrix Riccati equation. Plugging the solution for the optimal policy into the Bellman equation

$$x'Px = x'Rx + (-Fx)'Qu + 2(-Fx)'Hx + \beta(Ax + B(-Fx))'P(Ax + B(-Fx))$$

and rearranging, we get

$$x'Px = x'Rx+x'F'QFx-2x'F'Hx+\beta(x'A'PAx-x'A'PBFx-x'F'B'PAx+x'F'B'PBFx).$$

Since $x'A'PBFx$ is a scalar, it follows

$$x'A'PBFx = (x'A'PBFx)' = x'F'B'PAx$$

and therefore:

$$x'Px = x'Rx + x'F'QFx - 2x'F'Hx + \beta(x'A'PAx - 2x'F'B'PAx + x'F'B'PBFx),$$
$$x'Px = x'(R + F'QF - 2F'H + \beta(A'PA - 2F'B'PA + F'B'PBF))x.$$

From the last equation we obtain

$$P = R + F'QF - 2F'H + \beta(A'PA - 2F'B'PA + F'B'PBF),$$
$$P = R + \beta A'PA + F'(Q + \beta B'PB)F - 2F'(H + \beta B'PA).$$

Using the formula for F and simplifying

$$P = R+\beta A'PA+(H+\beta B'PA)'(Q+\beta B'PB)^{-1}(Q+\beta B'PB)(Q+\beta B'PB)^{-1}(H+\beta B'PA) -$$
$$- 2(H + \beta B'PA)'(Q + \beta B'PB)^{-1}(H + \beta B'PA) = R + \beta A'PA +$$
$$+ (H+\beta B'PA)'(Q+\beta B'PB)^{-1}(H+\beta B'PA)-2(H+\beta B'PA)'(Q+\beta B'PB)^{-1}(H+\beta B'PA) =$$
$$= R + \beta A'PA - (H + \beta B'PA)'(Q + \beta B'PB)^{-1}(H + \beta B'PA).$$

Problem 2. ***(LS 5.4a)*** *A household seeks to maximize*

$$-\sum_{t=1}^{\infty} \beta^t\{(s_t - b)^2 + \gamma i_t^2\}$$

subject to

$$c_t + i_t = ra_t + y_t,$$
$$a_{t+1} = a_t + i_t,$$
$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1},$$
$$s_t = \lambda h_t + \pi c_t,$$
$$h_{t+1} = \delta h_t + \theta c_t.$$

Here c_t, i_t, a_t, y_t, s_t, h_t are the household's consumption, investment, asset holdings, exogenous labor income, consumption services and stock of durables at t; $b > 0$, $\gamma > 0$, $r > 0$, $\beta \in (0,1)$ and ρ_1, ρ_2, λ, π, δ, θ are parameters, and y_0, y_{-1}, h_0 are initial conditions. Map this problem into an optimal linear regulator problem.

Remark: There are several solution to this problem. We will present two of the them: introducing an additional state variable, unity, and using the transformations in order to eliminate the constants.

Solution 1 (using transformations): First of all, let's choose the control and state variables:

- States: y_t, y_{t-1}, h_t, a_t
- Controls: s_t, c_t, a_{t+1}, h_{t+1}, i_t, y_{t+1}

Eliminating s_t, i_t and combining some constraints, we get a simplified setup

$$\max -\sum_{t=1}^{\infty} \beta^t \{(\lambda h_t + \pi c_t - b)^2 + \gamma(r a_t + y_t - c_t)^2\}$$

subject to

$$\begin{aligned} a_{t+1} &= a_t(r+1) + y_t - c_t, \\ y_{t+1} &= \rho_1 y_t + \rho_2 y_{t-1}, \\ h_{t+1} &= \delta h_t + \theta c_t. \end{aligned}$$

Notice that there is a constant in the objective function which does not allow us to map this problem into standard LQ setup. Therefore, we will "shift" our variables to get rid of the constants:

$$\begin{aligned} \hat{a}_t &= a_t - K_1, \\ \hat{c}_t &= c_t - K_2, \\ \hat{h}_t &= h_t - K_3, \end{aligned}$$

where K_1, K_2, K_3 are constants which need to be specified.

We need to get rid of constant in objective function. Express the term in the objective function in terms of transformed variables:

$$\lambda h_t + \pi c_t - b = \lambda(\hat{h}_t + K_3) + \pi(\hat{c}_t + K_2) - b = \lambda \hat{h}_t + \pi \hat{c}_t + \lambda K_3 + \pi K_2 - b.$$

Since our goal is to eliminate constant, we want the last term to sum up to zero. We obtain the following condition:

$$\lambda K_3 + \pi K_2 - b = 0. \tag{1}$$

Similarly, expressing the constraint in terms of transformed variables gives us next two conditions:

$$\hat{a}_{t+1} + K_1 = (\hat{a}_t + K_1)(r+1) + y_t - (\hat{c}_t + K_2) \quad \Rightarrow \quad K_1 = K_1(1+r) - K_2, \tag{2}$$

$$\hat{h}_{t+1} + K_3 = \delta(\hat{h}_t + K_3) + \theta(\hat{c}_t + K_2) \quad \Rightarrow \quad K_3 = \delta K_3 + \theta K_2. \tag{3}$$

Solving a linear system of the equations (1), (2), and (3), we obtain values of K_1, K_2, and K_3

$$K_1 = \frac{b(1-\delta)}{r(\lambda\theta + \pi(1-\delta))},$$

$$K_2 = \frac{b(1-\delta)}{\lambda\theta + \pi(1-\delta)},$$

$$K_3 = \frac{b\theta}{\lambda\theta + \pi(1-\delta)}.$$

Our transformed problem than becomes:

$$\max - \sum_{t=1}^{\infty} \beta^t \{(\lambda \hat{h}_t + \pi \hat{c}_t)^2 + \gamma(r\hat{a}_t + y_t - \hat{c}_t)^2\}$$

subject to

$$\hat{a}_{t+1} = \hat{a}_t(r+1) + y_t - \hat{c}_t,$$

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1},$$

$$\hat{h}_{t+1} = \delta \hat{h}_t + \theta \hat{c}_t.$$

Defining state vector $x_t = [\hat{a}_t, \hat{h}_t, y_t, y_{t-1}]'$ and vector of controls $u_t = \hat{c}_t$, we can easily map this problem into LQ setup:

$$R = \begin{pmatrix} -\gamma r^2 & 0 & -\gamma r & 0 \\ 0 & -\lambda^2 & 0 & 0 \\ -\gamma r & 0 & -\gamma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = -(\gamma + \pi^2), \quad H = \begin{pmatrix} \gamma r & -\lambda\pi & \gamma & 0 \end{pmatrix},$$

$$A = \begin{pmatrix} 1+r & 0 & 1 & 0 \\ 0 & \delta & 0 & 0 \\ 0 & 0 & \rho_1 & \rho_2 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ \theta \\ 0 \\ 0 \end{pmatrix}.$$

Solution 2 (introducing an additional state variable): In the previous solution, we have already derived a simplified setup to the problem

$$\max - \sum_{t=1}^{\infty} \beta^t \{(\lambda h_t + \pi c_t - b)^2 + \gamma(ra_t + y_t - c_t)^2\}$$

subject to

$$a_{t+1} = a_t(r+1) + y_t - c_t,$$

$$y_{t+1} = \rho_1 y_t + \rho_2 y_{t-1},$$

$$h_{t+1} = \delta h_t + \theta c_t.$$

Defining state vector $x_t = [a_t, h_t, y_t, y_{t-1}, 1]'$ and vector of controls $u_t = c_t$, we can easily map this problem into LQ setup:

$$R = \begin{pmatrix} -\gamma r^2 & 0 & -\gamma r & 0 & 0 \\ 0 & -\lambda^2 & 0 & 0 & \lambda b \\ -\gamma r & 0 & -\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -b^2 \end{pmatrix}, \quad Q = -(\gamma + \pi^2), \quad H = \begin{pmatrix} \gamma r & -\lambda\pi & \gamma & 0 & b\pi \end{pmatrix},$$

$$A = \begin{pmatrix} 1+r & 0 & 1 & 0 & 0 \\ 0 & \delta & 0 & 0 & 0 \\ 0 & 0 & \rho_1 & \rho_2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 \\ \theta \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$