Macro Topics: Introduction to Matlab
Fall 2016
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Exercise Session #6

Suggested Solutions

Problem 1. *(RBC Model)*

Consider the RBC model covered in class. Social planner maximizes utility of representative agent

$$\max \mathbb{E}_0 \sum_{t=0}^{\infty} b^t u(C_t, L_t), \quad b > 0,$$

where C_t is consumption, L_t is leisure. Agent is endowed with one unit of time, that can be allocated between work and leisure. Consumption good (Y_t) can be either consumed or invested (I_t). Production in the economy is given by Cobb–Douglas production function:

$$Y_t = A_t K_t^{1-\alpha}(N_t X_t)^{\alpha}$$

with capital K_t and labour N_t serving as inputs. A_t is productivity shock given by

$$A_t = A_{t-1}^{\rho} v_t, \quad \mathbb{E} v_t = 1.$$

X_t is the deterministic component of productivity and is assumed to be given by

$$X_{t+1} = \gamma X_t, \quad \gamma > 1.$$

Capital evolves according to

$$K_{t+1} = I_t + (1-\delta)K_t,$$

where δ is depreciation rate. Assume that K_0, X_0, and A_0 are given. Also assume that u has the following functional form

$$u(C_t, L_t) = \frac{(C_t \nu(L_t))^{1-\sigma} - 1}{1-\sigma}.$$

(1) Modify the model to eliminate steady–state growth.

(2) Identify state and control variables. Using the Bellman equation approach get the Euler equation of the economy.

(3) Log–linearize obtained equations (Euler equation, law of motion of capital, productivity law of motion) around steady–state using Taylor series approximation (for simplicity assume that there is no disutility from working).

Solution:

(1) In order to eliminate steady–state growth we divide all variables by X_t. Denote the new variables by small letters, that is

$$y_t = \frac{Y_t}{X_t}, \quad k_t = \frac{K_t}{X_t}, \quad i_t = \frac{I_t}{X_t}, \quad c_t = \frac{C_t}{X_t}.$$

Now we can also transform our equations:

$$\frac{Y_t}{X_t} = \frac{A_t K_t^{1-\alpha}(N_t X_t)^\alpha}{X_t} \quad \Rightarrow \quad y_t = A_t k_t^{1-\alpha} N_t^\alpha,$$

$$\frac{K_{t+1}}{X_t} = \frac{I_t + (1-\delta)K_t}{X_t} \quad \Rightarrow \quad \frac{K_{t+1}}{X_t}\frac{X_{t+1}}{X_{t+1}} = i_t + (1-\delta)k_t \quad \Rightarrow \quad \gamma k_{t+1} = i_t + (1-\delta)k_t,$$

$$\frac{Y_t}{X_t} = \frac{C_t + I_t}{X_t} \quad \Rightarrow \quad y_t = c_t + i_t.$$

We can also change utility function $u(C_t, L_t)$ to $u(c_t, L_t)$ by changing the discount factor to some β (for details see lecture notes).

The above equations can now be combined to one constraint

$$c_t = A_t k_t^{1-\alpha} N_t^\alpha + (1-\delta)k_t - \gamma k_{t+1},$$

plus we have time allocation constraint

$$N_t + L_t = 1.$$

(2) We take c_t, N_t, L_t, and k_{t+1} as control variables and k_t, A_t as state variables. We can eliminate c_t and L_t using the above equations. Now we can set up the Bellman equation:

$$V(k, A) = \max_{k', N}\{u(Ak^{1-\alpha}N^\alpha + (1-\delta)k - \gamma k', 1-N) + \beta \mathbb{E} V(k', A')\}.$$

First Order Conditions (derivatives with respect to k' and N) are

$$u_1(c, 1-N)(-\gamma) + \beta \mathbb{E} V_1(k', A') = 0,$$

$$u_1(c, 1-N)(A\alpha k^{1-\alpha}N^{\alpha-1}) - u_2(c, 1-N) = 0,$$

where

$$c = Ak^{1-\alpha}N^\alpha + (1-\delta)k - \gamma k',$$

and Envelope Theorem condition is (derivative with respect to k only, A is predetermined state):

$$V_1(k, A) = u_1(c, 1-N)(A(1-\alpha)k^{-\alpha}N^\alpha + 1 - \delta).$$

To get the Euler Equation we first shift ET condition one period forward to get

$$V_1(k', A') = u_1(c', 1-N')(A'(1-\alpha)(k')^{-\alpha}(N')^\alpha + 1 - \delta)$$

and plug it in FOC for k':

$$-\gamma u_1(c, 1-N) + \beta \mathbb{E}[u_1(c', 1-N')(A'(1-\alpha)(k')^{-\alpha}(N')^\alpha + 1 - \delta)] = 0.$$

(3) From now on assume that there is no disutility from labour, that is $N_t = 1$ and normalize $\nu(L_t) = 1$, $\forall L_t$.

Therefore we have the following 3 equations we need to log–linearize:

$$\gamma c_t^{-\sigma} = \beta \mathbb{E}_t[c_{t+1}^{-\sigma}(A_{t+1}(1-\alpha)k_{t+1}^{-\alpha} + 1 - \delta)], \tag{1}$$

$$c_t = A_t k_t^{1-\alpha} + (1-\delta)k_t - \gamma k_{t+1}, \tag{2}$$

$$A_t = A_{t-1}^{\rho} v_t. \tag{3}$$

Our task is to write equations in terms of

$$\tilde{c}_t = \frac{c_t - \bar{c}}{\bar{c}}, \quad \tilde{A}_t = \frac{A_t - \bar{A}}{\bar{A}}, \quad \tilde{k}_t = \frac{k_t - \bar{k}}{\bar{k}}, \quad \tilde{v}_t = \frac{v_t - \bar{v}}{\bar{v}}.$$

We start with the last one.

Productivity law of motion.

We need to use Taylor series expansion only for the right–hand side of the equation, since the left–hand side is already linear. First we evaluate the equation in steady–state:

$$\bar{A} = \bar{A}^{\rho}\bar{v}.$$

Then, using Taylor formula around steady–state we get:

$$A_{t-1}^{\rho} v_t = \bar{A}^{\rho}\bar{v} + \rho\bar{A}^{\rho-1}\bar{v}(A_{t-1} - \bar{A}) + \bar{A}^{\rho}(v_t - \bar{v}).$$

Equation (3) we can write as

$$A_t - \bar{A} = A_{t-1}^{\rho} v_t - \bar{A}^{\rho}\bar{v} \quad \Rightarrow \quad A_t - \bar{A} = \rho\bar{A}^{\rho-1}\bar{v}(A_{t-1} - \bar{A}) + \bar{A}^{\rho}(v_t - \bar{v})$$

or rearranging we get

$$\bar{A}\tilde{A}_t = \rho\bar{A}^{\rho}\bar{v}\tilde{A}_{t-1} + \bar{A}^{\rho}\bar{v}\tilde{v}_t \quad \Rightarrow \quad \tilde{A}_t = \rho\tilde{A}_{t-1} + \tilde{v}_t.$$

Feasibility constraint.

We start by putting equation (2) into the steady state:

$$\bar{c} = \bar{A}\bar{k}^{1-\alpha} + (1-\delta)\bar{k} - \gamma\bar{k}$$

and subtract $\bar{c}$ from both sides of equation (2) to get:

$$c_t - \bar{c} = (A_t k_t^{1-\alpha} - \bar{A}\bar{k}^{1-\alpha}) + (1-\delta)(k_t - \bar{k}) - \gamma(k_{t+1} - \bar{k}).$$

The only non–linear part is $A_t k_t^{1-\alpha}$, so we use Taylor series approximation around the steady–state:

$$A_t k_t^{1-\alpha} = \bar{A}\bar{k}^{1-\alpha} + \bar{k}^{1-\alpha}(A_t - \bar{A}) + (1-\alpha)\bar{A}\bar{k}^{-\alpha}(k_t - \bar{k})$$

and plug it in the above equation

$$c_t - \bar{c} = \bar{k}^{1-\alpha}(A_t - \bar{A}) + (1-\alpha)\bar{A}\bar{k}^{-\alpha}(k_t - \bar{k}) + (1-\delta)(k_t - \bar{k}) - \gamma(k_{t+1} - \bar{k})$$

or

$$c_t - \bar{c} = \bar{k}^{1-\alpha}(A_t - \bar{A}) + ((1-\alpha)\bar{A}\bar{k}^{-\alpha} + 1 - \delta)(k_t - \bar{k}) - \gamma(k_{t+1} - \bar{k}).$$

Now we get:

$$\tilde{c}_t = \bar{k}^{1-\alpha}\frac{\bar{A}}{\bar{c}}\tilde{A}_t + \frac{\bar{k}}{\bar{c}}((1-\alpha)\bar{A}\bar{k}^{-\alpha} + 1 - \delta)\tilde{k}_t - \gamma\frac{\bar{k}}{\bar{c}}\tilde{k}_{t+1}.$$

Euler Equation.

Since both sides of the Euler Equation are non–linear we will log–linearize both sides separately. As usual we will start by rewriting the equation in steady state:

$$\gamma\bar{c}^{-\sigma} = \beta\mathbb{E}[\bar{c}^{-\sigma}((1-\alpha)\bar{A}\bar{k}^{-\alpha}+1-\delta))],$$

and subtracting from both sides of (1) to get:

$$\gamma c_t^{-\sigma} - \gamma\bar{c}^{-\sigma} = \beta\mathbb{E}_t[c_{t+1}^{-\sigma}(A_{t+1}(1-\alpha)k_{t+1}^{-\alpha}+1-\delta) - \bar{c}^{-\sigma}((1-\alpha)\bar{A}\bar{k}^{-\alpha}+1-\delta)].$$

Now, the LHS of the above equation we can write (using Taylor series approximation) as:

$$\gamma c_t^{-\sigma} - \gamma\bar{c}^{-\sigma} = -\gamma\sigma\bar{c}^{-\sigma-1}(c_t-\bar{c}) = -\gamma\sigma\bar{c}^{-\sigma}\tilde{c}_t.$$

Omitting for now the expectation sign, the RHS becomes:

$$c_{t+1}^{-\sigma}\underbrace{(A_{t+1}(1-\alpha)k_{t+1}^{-\alpha}+1-\delta)}_{a_{t+1}} - \bar{c}^{-\sigma}\underbrace{((1-\alpha)\bar{A}\bar{k}^{-\alpha}+1-\delta)}_{\bar{a}} =$$

$$= -\sigma\bar{c}^{-\sigma}\bar{a}\tilde{c}_{t+1}+\bar{c}^{-\sigma}(a_{t+1}-\bar{a}) = -\sigma\bar{c}^{-\sigma}\bar{a}\tilde{c}_{t+1}+\bar{c}^{-\sigma}((1-\alpha)\bar{k}^{-\alpha}\bar{A}\tilde{A}_{t+1}-\alpha(1-\alpha)\bar{k}^{-\alpha}\bar{A}\tilde{k}_{t+1}).$$

Therefore the Euler Equation becomes

$$-\gamma\sigma\tilde{c}_t = \beta\mathbb{E}_t[-\sigma\bar{a}\tilde{c}_{t+1}+(1-\alpha)\bar{k}^{-\alpha}\bar{A}\tilde{A}_{t+1}-\alpha(1-\alpha)\bar{k}^{-\alpha}\bar{A}\tilde{k}_{t+1}].$$

Problem 2. ***(Uhlig's Method)***

Consider the problem of representative agent in the economy

$$\max\sum_{t=0}^{\infty}\beta^t\left(\frac{C_t^{1-\eta}}{1-\eta}-\frac{H_t^{1-\phi}}{1-\phi}\right)$$

subject to budget constraint

$$C_t = \lambda_t K_t^{\theta}H_t^{1-\theta}+(1-\delta)K_t-K_{t+1},$$

where C_t is consumption, H_t is labour input, K_t is capital (δ is the rate of depreciation of capital).

(1) Derive the FOCs, ET, and EE for the problem.

(2) Log–linearize obtained FOCs/EE and budget constraint using Uhlig's Method.

Solution:

(1) We take C_t, H_t, and K_{t+1} as control variables and K_t as a state variable. The Bellman equation is the following:

$$V(K_t) = \max_{H_t,K_{t+1}}\left\{\frac{(\lambda_t K_t^{\theta}H_t^{1-\theta}+(1-\delta)K_t-K_{t+1})^{1-\eta}}{1-\eta}-\frac{H_t^{1-\phi}}{1-\phi}+\beta V(K_{t+1})\right\}.$$

First Order Conditions (derivatives with respect to K_{t+1} and H_t) are

$$-C_t^{-\eta}+\beta V'(K_{t+1}) = 0,$$

$$C_t^{-\eta}(1-\theta)\lambda_t K_t^{\theta} H_t^{-\theta} - H_t^{-\phi} = 0,$$

where

$$C_t = \lambda_t K_t^{\theta} H_t^{1-\theta} + (1-\delta)K_t - K_{t+1},$$

and Envelope Theorem condition is (derivative with respect to K_t):

$$V'(K_t) = C_t^{-\eta}(\lambda_t \theta K_t^{\theta-1} H_t^{1-\theta} + 1 - \delta).$$

To get the Euler Equation we first shift ET condition one period forward to get

$$V'(K_{t+1}) = C_{t+1}^{-\eta}(\lambda_{t+1}\theta K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + 1 - \delta)$$

and plug it in FOC for K_{t+1}:

$$-C_t^{-\eta} + \beta C_{t+1}^{-\eta}(\lambda_{t+1}\theta K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + 1 - \delta) = 0.$$

(2) Denote by

$$\tilde{X}_t = \log X_t - \log \bar{X},$$

for some X_t with steady–state value $\bar{X}$. Then

$$X_t = \bar{X} e^{\tilde{X}_t}.$$

In order to log–linearize equations we will use the following rule of Uhlig's method:

$$e^{\tilde{X}_t + \alpha \tilde{Y}_t} \approx 1 + \tilde{X}_t + \alpha \tilde{Y}_t.$$

We have the following FOCs and EE:

$$\beta[\theta \lambda_{t+1} K_{t+1}^{\theta-1} H_{t+1}^{1-\theta} + 1 - \delta] C_{t+1}^{-\eta} = C_t^{-\eta}$$

and

$$C_t^{-\eta}(1-\theta)\lambda_t K_t^{\theta} H_t^{-\theta} = H_t^{-\phi}.$$

Also we have budget constraint given by:

$$C_t = \lambda_t K_t^{\theta} H_t^{1-\theta} + (1-\delta)K_t - K_{t+1}.$$

Applying Uhlig's method to the first equation we get

$$\beta[\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta} e^{\tilde{\lambda}_{t+1} + (\theta-1)\tilde{K}_{t+1} + (1-\theta)\tilde{H}_{t+1}} + 1 - \delta] = e^{\eta \tilde{C}_{t+1} - \eta \tilde{C}_t},$$

which can be approximated as

$$\beta[\theta \bar{\lambda} \bar{K}^{\theta-1} \bar{H}^{1-\theta}(1 + \tilde{\lambda}_{t+1} + (\theta-1)\tilde{K}_{t+1} + (1-\theta)\tilde{H}_{t+1}) + 1 - \delta] = 1 + \eta \tilde{C}_{t+1} - \eta \tilde{C}_t.$$

Similarly, the second equation we transform to

$$(1-\theta)\bar{\lambda} \bar{K}^{\theta} \bar{H}^{-\theta} e^{\tilde{\lambda}_t + \theta \tilde{K}_t - \theta \tilde{H}_t} = \frac{\bar{C}^{\eta}}{\bar{H}^{\phi}} e^{\eta \tilde{C}_t - \phi \tilde{H}_t},$$

which we can approximate as

$$(1-\theta)\bar{\lambda} \bar{K}^{\theta} \bar{H}^{-\theta}(1 + \tilde{\lambda}_t + \theta \tilde{K}_t - \theta \tilde{H}_t) = \frac{\bar{C}^{\eta}}{\bar{H}^{\phi}}(1 + \eta \tilde{C}_t - \phi \tilde{H}_t).$$

Finally, the budget constraint becomes

$$\bar{C} e^{\tilde{C}_t} = \bar{\lambda} \bar{K}^{\theta} \bar{H}^{1-\theta} e^{\tilde{\lambda}_t + \theta \tilde{K}_t + (1-\theta)\tilde{H}_t} + (1-\delta)\bar{K} e^{\tilde{K}_t} - \bar{K} e^{\tilde{K}_{t+1}}$$

or

$$\bar{C}(1 + \tilde{C}_t) = \bar{\lambda} \bar{K}^{\theta} \bar{H}^{1-\theta}(1 + \tilde{\lambda}_t + \theta \tilde{K}_t + (1-\theta)\tilde{H}_t) + (1-\delta)\bar{K}(1 + \tilde{K}_t) - \bar{K}(1 + \tilde{K}_{t+1}).$$