Macro Topics: Introduction to Matlab
Fall 2016
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Exercise Session #2

Suggested Solutions

Problem 1. ***(Habit Persistence)*** *Consider the problem of consumer who seeks to solve*

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\log(c_t) + \gamma \log(c_{t-1})) \quad \textit{with } \ 0 < \gamma < 1$$

subject to the following constraints:

$$k_{t+1} + c_t \leq A k_t^{\alpha}, \quad A > 0, \ \ \alpha \in (0,1),$$

$$c_t > 0, \ \ k_t > 0, \ \ k_0, c_{-1} \textit{ given.}$$

(1) Clearly identify state and control variable(s).
(2) Set up the Bellman Equation for the problem (that is write the problem in the recursive form). ***Hint:*** *Value function for the consumer will be the function of two variables.*
(3) Derive FOC, ET and EE.
(4) Derive expression for steady state level of capital and consumption in terms of parameters of the model.

Solution:

(1) The state and control variables are:
- states: k_t, c_{t-1};
- controls: c_t, k_{t+1}.

(2) The Bellman Equation has the following form:

$$V(k, c_{-1}) = \max_{\tilde{k}, c} \{u(c) + \beta V(\tilde{k}, c)\} \ \text{ s.t } \ c = Ak^{\alpha} - \tilde{k}.$$

Or after substitution:

$$V(k, c_{-1}) = \max_{\tilde{k}} \{u(Ak^{\alpha} - \tilde{k}) + \beta V(\tilde{k}, c)\} = \max_{\tilde{k}} \{\log(c) + \gamma \log(c_{-1}) + \beta V(\tilde{k}, c)\}.$$

(3) We proceed as usually:

$$FOC: \quad -\frac{1}{c} + \beta[V_1(\tilde{k}, c) - V_2(\tilde{k}, c)] = 0,$$

$$\text{ET w.r.t. } k: \quad \frac{\partial V(k, c_{-1})}{\partial k} = \frac{1}{c}\alpha A k^{\alpha - 1} + \beta \frac{\partial V(\tilde{k}, c)}{\partial c} \alpha A k^{\alpha - 1},$$

$$\text{ET w.r.t. } c_{-1}: \quad \frac{\partial V(k, c_{-1})}{\partial c_{-1}} = \gamma \frac{1}{c_{-1}}.$$

Since we have partial of future value function w.r.t. c in the first envelope condition, we have to substitute it there (after forwarding it one period) before substituting that envelope condition into FOC:

$$\frac{\partial V(k, c_{-1})}{\partial k} = \frac{1}{c}\alpha A k^{\alpha-1} + \beta\gamma\frac{1}{c}\alpha A k^{\alpha-1}.$$

Substituting both envelope conditions (in correct timing) into the FOC yields:

$$\frac{1}{c} = \beta\left\{\frac{1}{\tilde{c}}\alpha A\tilde{k}^{\alpha-1} + \beta\gamma\frac{1}{\tilde{c}}\alpha A\tilde{k}^{\alpha-1} - \gamma\frac{1}{c}\right\}.$$

Now rearranging to obtain c on the left hand side and $\tilde{c}$ on the right hand side we obtain Euler Equation:

$$\frac{1}{c}(1+\beta\gamma) = (1+\beta\gamma)\beta\frac{1}{\tilde{c}}\alpha A\tilde{k}^{\alpha-1},$$

$$\frac{1}{c} = \beta\frac{1}{\tilde{c}}\alpha A\tilde{k}^{\alpha-1}.$$

(4) Now to derive the steady state values of capital and consumption in terms of the parameters of the model, we have to set $c = \tilde{c} = \bar{c}$ and $k = \tilde{k} = \bar{k}$ and plug them into Euler Equation and the budget constraint:

$$\frac{1}{\bar{c}} = \beta\frac{1}{\bar{c}}\alpha A\bar{k}^{\alpha-1} \quad \Rightarrow \quad \bar{k} = (A\alpha\beta)^{1/(1-\alpha)}.$$

For consumption we use the transiton law:

$$\bar{c} = A\bar{k}^{\alpha} - \bar{k} = (A\alpha\beta)^{1/(1-\alpha)}\left(\frac{1}{\alpha\beta} - 1\right).$$

Problem 2. *Consider social planner's problem of maximizing lifetime utility of the representative consumer:*

$$\sum_{t=0}^{\infty}\beta^t u(c_t, 1-l_t)$$

subject to the following constraints:

$$y_t = c_t + i_t,$$

$$y_t = f(k_t, l_t),$$
$$k_{t+1} = (1-\delta)k_t + i_t,$$
$$k_{t+1} \geq 0, \quad c_t \geq 0,$$
$$k_0 \text{ is given},$$

where c is consumption, i is investment, k is capital, l is labour input, and output y is produced from capital and labour using production function $f(\cdot,\cdot)$, δ is the depreciation rate of capital.

(1) Clearly identify state and control variables. Set up the Bellman Equation for the problem (that is write the problem in the recursive form).

(2) Derive First Order Conditions and Envelope Theorem conditions.

(3) Using the equations from above derive Euler Equation and calculate steady state values of c, k, and l.

Solution:

(1) The state and control variables are:

- states: k_t;
- controls: c_t, i_t, k_{t+1}, l_t.

We can combine given constraints into one to get:

$$c_t = f(k_t, l_t) - k_{t+1} + (1-\delta)k_t.$$

Now objective function transforms to

$$\sum_{t=0}^{\infty} \beta^t u(f(k_t, l_t) - k_{t+1} + (1-\delta)k_t, 1 - l_t).$$

Denote $k_t = k$, $k_{t+1} = \tilde{k}$, $l_t = l$, and $l_{t+1} = \tilde{l}$. We can set up the Bellman Equation:

$$V(k) = \max_{\tilde{k}, l} \{u(f(k,l) + (1-\delta)k - \tilde{k}, 1 - l) + \beta V(\tilde{k})\}.$$

(2) In order to simplify notation, denote by $u_i(x_1, ..., x_n) = {}^{\partial u(x_1,...,x_n)}/_{\partial x_i}$, $i = 1...n$ for some function u.

Then, FOC for $\tilde{k}$ becomes

$$-u_1(f(k,l) + (1-\delta)k - \tilde{k}, 1-l) + \beta V'(\tilde{k}) = 0$$

and for l

$$u_1(f(k,l) + (1-\delta)k - \tilde{k}, 1-l) \cdot f_2(k,l) - u_2(c, 1-l) = 0.$$

ET condition (the derivative of the Bellman Equation w.r.t. k) is

$$V'(k) = u_1(f(k,l) + (1-\delta)k - \tilde{k}, 1-l) \cdot (f_1(k,l) + 1 - \delta).$$

(3) We can shift the last equation one period ahead and plug into the first FOC. After rearranging we get Euler Equation:

$$\frac{u_1(f(k,l) + (1-\delta)k - \tilde{k}, 1-l)}{u_1(f(\tilde{k},\tilde{l}) + (1-\delta)\tilde{k} - \tilde{\tilde{k}}, 1-\tilde{l})} = \beta(f_1(\tilde{k},\tilde{l}) + 1 - \delta).$$

In steady state we have $l = \tilde{l} = \bar{l}$, $k = \tilde{k} = \tilde{\tilde{k}} = \bar{k}$, and $c = \bar{c}$. We can use the following equations to determine the steady state values:

$$f_1(\bar{k},\bar{l}) + 1 - \delta = \frac{1}{\beta},$$

$$\bar{c} + \delta\bar{k} = f(\bar{k},\bar{l}),$$

$$u_1(\bar{c}, 1-\bar{l}) \cdot f_2(\bar{k},\bar{l}) - u_2(\bar{c}, 1-\bar{l}) = 0.$$