Macro Topics: Introduction to Matlab
Spring Semester 2019/2020
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# Midterm Exam

*You have 2 hours to complete this setup. The exam is worth 100 points in total, exact amount of points is indicated for each exercise. Any violation of academic integrity will be punished accordingly.*

**Problem 1.** (**30 points**) Consider the modification of Inventory Control problem, where

- $x_k$ — stock of a good available at the beginning of $k$–th period
- $u_k$ — stock ordered at the beginning of $k$–th period
- $w_k$ — demand during the $k$–th period (possibly random variable)

The stock evolves according to the discrete time equation:

$$x_{k+1} = x_k - u_k + w_k, \quad k = 0, 1, ..., N.$$

The purchasing costs in every period are equal $x_k u_k$ and penalty for non–zero stock is equal $(x_k - u_k + w_k)^2$, thus, costs incurred in period $k$ are sum of those two. Suppose that there are no costs at the last period, i.e. $g(x_N) = 0$. Suppose also that $w_k$ is a random variable with mean 1 and variance 2 ($\mathbb{E}[w_k] = 1$, $\mathbb{E}[w_k^2] - (\mathbb{E}[w_k])^2 = 2$).

Solve the problem for $N = 2$, i.e. find optimal policies in both periods and cost–to–go functions (which should look as second–order polynomials).

**Problem 2.** Consider the following problem of the optimal harvesting of the renewable resource. A social planner wants to maximize the present value of the utility of the representative consumer

$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to

$$Z_{t+1} = Z_t + M(1 - e^{-KZ_t}) - S_t,$$
$$Z_t \geq 0, \quad S_t \geq 0,$$

where $Z_t$ is the current stock of renewable resource and $S_t$ is the harvesting rate. $K$ and $M$ are some positive constants. There is no uncertainty and no population growth. Representative consumer receives utility from consumption $C_t$. Non–storable consumption good is produced using $S_t$ as the only input:

$$C_t = Y_t = S_t^{\alpha}, \quad \alpha < 1.$$

Utility function $u$ is twice differentiable and concave.

(a) (**5 points**) Clearly identify state and control variables. Rewrite the problem in the following form:

$$\max_{x_t} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}),$$

where $x_t$ is the state variable.

(b) (**5 points**) Write Bellman Equation for the problem.

(c) (**10 points**) Derive FOC(s) and the Envelope Theorem condition(s).

(d) (**10 points**) Substitute ET(s) into FOC(s) and obtain Euler Equation for the problem.

(e) (**10 points**) Derive the steady state value of stock of resources $Z^*$ and extraction rate $S^*$. What happens to $Z^*$ and $S^*$ as representative agent becomes more (higher $\beta$) or less (lower $\beta$) "patient"?

**Problem 3.** (**30 points**) Consider a consumer with utility function

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}, \quad 0 < \beta < 1, \quad \sigma > 0.$$

The consumer is endowed with a cake of size $x_0$ at time $t = 0$. Each period, she has a cake $x_t$ and can either consume some, $c_t$, or hold some cake over to the next period, $x_{t+1}$.

Guess that the value function $V(x_t)$ is of the following form:

$$V(x_t) = \alpha \frac{x_t^{1-\sigma}}{1-\sigma},$$

for some unknown coefficient $\alpha > 0$. Solve for the unknown coefficient. Given your value function, calculate the optimal policy. (*Hint:* You will need to derive both FOC and ET to solve this problem.)