Macro Topics: Introduction to Matlab
Fall 2016
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Final Exam

Suggested Solutions

Problem 1. ***(Wage growth rate and the reservation wage)***

An unemployed worker receives each period an offer to work for wage w_t forever, where $w_t = w$ in the first period and $w_t = \phi^t w$ after t periods on the job. Assume $\phi > 1$, that is, wages increase with tenure. The initial wage offers are independently and identically drawn from the cumulative probability distribution function $F(w)$, with $F(0) = 0$, $F(B) = 1$ for $B < 1$.

The worker's objective function is to maximize

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t y_t, \quad \textit{where } 0 < \beta < 1,$$

and $y_t = w_t$ if the worker is employed and $y_t = c$ if the worker is unemployed, where c is unemployment compensation. Assume that $\phi\beta < 1$.

(a) ***(10 points)*** *Let $V(w)$ be the optimal value of the objective function for an unemployed worker who has offer w in hand. Write the Bellman equation for this problem.*

(b) ***(20 points)*** *Derive the reservation wage for this problem. (Hint: Substitute the expression for $V(w)$ into the reservation wage equation.)*

Solution:

(a) If the worker accepts the employment at wage w, the sequence $\{y_t\}$ is given by $y_0 = w$, $y_1 = \phi w$, ..., $y_t = \phi^t w$, ... Therefore, the value of the objective function if the worker accepts the wage offer is

$$\sum_{t=0}^{\infty}\beta^t y_t = \sum_{t=0}^{\infty}\beta^t\phi^t w = \frac{w}{1-\beta\phi}.$$

The Bellman equation for the worker's problem is

$$V(w) = \max\left\{\frac{w}{1-\beta\phi},\ c + \beta\int_0^B V(w')dF(w')\right\}.$$

(b) The reservation wage satisfies

$$\frac{\bar{w}}{1-\beta\phi} = c + \beta\int_0^B V(w')dF(w').$$

The optimal policy is to accept the wage offer with an initial wage higher than a reservation wage $\bar{w}$. Therefore, the value function $V(w)$ is given by

$$V(w) = \begin{cases} \frac{\bar{w}}{1-\beta\phi} & \text{if } w \leq \bar{w}, \\ \frac{w}{1-\beta\phi} & \text{if } w \geq \bar{w}. \end{cases}$$

Substituting for $V(w)$ its expression, we can write the reservation wage equation as follows:

$$\frac{\bar{w}}{1-\beta\phi} = c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta\phi} dF(w') + \beta \int_{\bar{w}}^{B} \frac{w'}{1-\beta\phi} dF(w').$$

Rearranging, we obtain:

$$(1-\beta)\bar{w} - \beta \int_{\bar{w}}^{B} (w' - \bar{w}) dF(w') = (1-\beta\phi)c.$$

Problem 2. ***(Log–linearization)***

Consider the problem of representative agent in the economy

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}$$

subject to budget constraint

$$c_t + k_{t+1} - (1-\delta)k_t = k_t^{\alpha}, \quad k_0 \text{ given},$$

where c_t is consumption, k_t is capital (δ is the rate of depreciation of capital).

(a) ***(5 points)*** *Identify state and control variables. Write down the Bellman equation for the problem.*

(b) ***(15 points)*** *Derive First Order Condition, Envelope Theorem Condition, and Euler Equation for the problem.*

(c) ***(15 points)*** *Log–linearize the budget constraint using Taylor series approximation.*

(d) ***(15 points)*** *Use Uhlig's method to log–linearize the Euler Equation.*

Solution:

(a)
- **states:** k_t,
- **controls:** c_t, k_{t+1}.

The Bellman equation is (we will keep c_t in the equation for now, but we actually eliminate it using budget constraint):

$$V(k_t) = \max_{k_{t+1}} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta V(k_{t+1}) \right\},$$

where $c_t = k_t^{\alpha} + (1-\delta)k_t - k_{t+1}$.

(b) FOC (derivative with respect to k_{t+1}) is:

$$-c_t^{-\sigma} + \beta V'(k_{t+1}) = 0.$$

ET condition (derivative with respect to k_t) is:

$$V'(k_t) = c_t^{-\sigma}(\alpha k_t^{\alpha-1} + 1 - \delta).$$

To get the Euler Equation we shift ET condition one period ahead

$$V'(k_{t+1}) = c_{t+1}^{-\sigma}(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)$$

and substitute it to the FOC:

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}(\alpha k_{t+1}^{\alpha-1} + 1 - \delta).$$

(c) Now we can proceed with log–linearization of budget constraint:

$$c_t + k_{t+1} - (1-\delta)k_t = k_t^\alpha.$$

In the steady state the equation becomes

$$\bar{c} + \bar{k} - (1-\delta)\bar{k} = \bar{k}^\alpha.$$

Subtracting this expression from both sides of the equation above we get

$$c_t - \bar{c} + k_{t+1} - \bar{k} - (1-\delta)(k_t - \bar{k}) = k_t^\alpha - \bar{k}^\alpha.$$

We start by linearizing roght side of the equation. Using Taylor series approximation around steady–state we get:

$$k_t^\alpha - \bar{k}^\alpha = \alpha \bar{k}^{\alpha-1}(k_t - \bar{k}).$$

Our task is to write budget constraint in terms of

$$\tilde{c}_t = \frac{c_t - \bar{c}}{\bar{c}},\ \tilde{k}_t = \frac{k_t - \bar{k}}{\bar{k}},\ \tilde{k}_{t+1} = \frac{k_{t+1} - \bar{k}}{\bar{k}}.$$

Substituting the above expressions, we obtain

$$\bar{c}\tilde{c}_t + \bar{k}\tilde{k}_{t+1} - (1-\delta)\bar{k}\tilde{k}_t = \alpha \bar{k}^\alpha \tilde{k}_t.$$

(d) Our task is to write the Euler Equation

$$c_t^{-\sigma} = \beta c_{t+1}^{-\sigma}(\alpha k_{t+1}^{\alpha-1} + 1 - \delta)$$

in terms of

$$c_t = \bar{c}e^{\tilde{c}_t},\ c_{t+1} = \bar{c}e^{\tilde{c}_{t+1}},\ k_{t+1} = \bar{k}e^{\tilde{k}_{t+1}}.$$

Substituting the above expressions, we obtain

$$\bar{c}^{-\sigma}e^{-\sigma\tilde{c}_t} = \beta \bar{c}^{-\sigma}e^{-\sigma\tilde{c}_{t+1}}(\alpha \bar{k}^{\alpha-1}e^{(\alpha-1)\tilde{k}_{t+1}} + 1 - \delta).$$

Rearranging,

$$e^{-\sigma\tilde{c}_t} = \beta(\alpha \bar{k}^{\alpha-1}e^{(\alpha-1)\tilde{k}_{t+1} - \sigma\tilde{c}_{t+1}} + (1-\delta)e^{-\sigma\tilde{c}_{t+1}}).$$

Using approximation $e^x \approx 1 + x$, we get

$$1 - \sigma\tilde{c}_t = \beta(\alpha \bar{k}^{\alpha-1}(1 + (\alpha-1)\tilde{k}_{t+1} - \sigma\tilde{c}_{t+1}) + (1-\delta)(1 - \sigma\tilde{c}_{t+1})).$$

Using EE in steady states

$$1 = \beta(\alpha \bar{k}^{\alpha-1} + 1 - \delta),$$

we get

$$-\sigma\tilde{c}_t = \beta(\alpha \bar{k}^{\alpha-1}((\alpha-1)\tilde{k}_{t+1} - \sigma\tilde{c}_{t+1}) - (1-\delta)\sigma\tilde{c}_{t+1}).$$

Problem 3. ***(LQ problem)*** *(**20 points**)*

Consider the following problem of a Central Bank which tries to minimize the following loss function:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t (\pi_t^2 + \lambda U_t^2).$$

Here π_t is the inflation rate and U_t the unemployment rate (or, to be precise, deviation of the unemployment from its natural rate), $\lambda > 0$. The Central Bank believes that the unemployment rate evolves as

$$U_{t+1} = \pi_t - \gamma \pi_{t-1} + \rho U_t + \delta U_{t-1}.$$

Parameters of the model are such that the state transition equation is stable. The Bank can directly control inflation rate π_t.

Set up the problem in the standard LQ from.

Solution:

Since Central Bank can control inflation, we will take π_t as control variable, and U_t, U_{t-1}, and π_{t-1} as state variables. Denote by x_t current state variable and by u_t control variable:

$$x_t = \begin{pmatrix} U_t \\ U_{t-1} \\ \pi_{t-1} \end{pmatrix}, \quad u_t = \pi_t.$$

We start by rewriting constraint as

$$x_{t+1} = Ax_t + Bu_t,$$

where A and B are some matrices. We have:

$$\underbrace{\begin{pmatrix} U_{t+1} \\ U_t \\ \pi_t \end{pmatrix}}_{x_{t+1}} = \underbrace{\begin{pmatrix} \rho & \delta & -\gamma \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} U_t \\ U_{t-1} \\ \pi_{t-1} \end{pmatrix}}_{x_t} + \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}_{B} \underbrace{\pi_t}_{u_t}.$$

Now we can rewrite objective function in the form:

$$x_t' R x_t + u_t' Q u_t + 2u_t' H x_t.$$

We have

$$\pi_t^2 + \lambda U_t^2 = x_t' \underbrace{\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{-R} x_t + u_t' \underbrace{1}_{-Q} u_t + 2u_t' \underbrace{\begin{pmatrix} 0 & 0 & 0 \end{pmatrix}}_{H} x_t.$$