Macro Topics: Introduction to Matlab
Fall 2016
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Midterm Exam

Suggested Solutions

Problem 1. *Consider the modification of Inventory Control problem, where*

- x_k *— stock of a good available at the beginning of k–th period*
- u_k *— stock ordered at the beginning of k–th peroiod*
- w_k *— demand during the k–th period (possibly random variable)*

The stock evolves according to the discrete time equation:

$$x_{k+1} = x_k - u_k + w_k, \quad k = 0, 1, ..., N.$$

The purchasing costs in every period are equal $x_k u_k$ and penalty for non–zero stock is equal $(x_k - u_k + w_k)^2$, thus, costs incurred in period k are sum of those two. Suppose that there are no costs at the last period, i.e. $g(x_N) = 0$. Suppose also that w_k is a random variable with mean 1 and variance 2 ($\mathbb{E}[w_k] = 1$, $\mathbb{E}[w_k^2] - (\mathbb{E}[w_k])^2 = 2$).

Solve the problem for $N = 2$, i.e. find optimal policies in both periods and cost–to–go functions (which should look as second–order polynomials).

Solution: Since there is no cost in the last period, cost–to–go function in the end of the second period $J_2(x_2) = 0$. Using the principle of optimality, consider the problem, where firm solves the following problem in the beginning of the period one:

$$J_1(x_1) = \min_{u_1} \mathbb{E}_{w_1}\{g_1(x_1, u_1) + J_2(x_2)\} = \min_{u_1} \mathbb{E}_{w_1}\{g_1(x_1, u_1)\} =$$

$$= \min_{u_1} \mathbb{E}_{w_1}\{x_1 u_1 + (x_1 - u_1 + w_1)^2\} = \min_{u_1}\{x_1 u_1 + (x_1 - u_1)^2 + 2(x_1 - u_1)\mathbb{E}(w_1) + \mathbb{E}(w_1^2)\} =$$

$$= \min_{u_1}\{x_1 u_1 + (x_1 - u_1)^2 + 2(x_1 - u_1) + 3\}.$$

Taking FOC with respect to u_1 (assuming u_1 can be any real number), we get:

$$x_1 - 2(x_1 - u_1) - 2 = 0,$$

and after some rearranging

$$u_1 = 1 + \frac{x_1}{2}.$$

Plugging this policy function back into the firm's problem, we obtain:

$$J_1(x_1) = x_1\left(1 + \frac{x_1}{2}\right) + \left(x_1 - 1 - \frac{x_1}{2}\right)^2 + 2\left(x_1 - 1 - \frac{x_1}{2}\right) + 3 = \frac{3}{4}x_1^2 + x_1 + 2.$$

Now we have cost–to–go function for the period one and we can solve the firm's problem in the period zero:

$$J_0(x_0) = \min_{u_0} \mathbb{E}_{w_0}\{g_0(x_0, u_0) + J_1(x_1)\} = \min_{u_0} \mathbb{E}_{w_0}\{x_0 u_0 + (x_0 - u_0 + w_0)^2 + J_1(x_0 - u_0 + w_0)\} =$$

$$= \min_{u_0} \mathbb{E}_{w_0} \left\{ x_0 u_0 + (x_0 - u_0 + w_0)^2 + \frac{3}{4}(x_0 - u_0 + w_0)^2 + x_0 - u_0 + w_0 + 2 \right\} =$$

$$= \min_{u_0} \left\{ x_0 u_0 + \frac{7}{4}(x_0 - u_0)^2 + \frac{9}{2}(x_0 - u_0) + \frac{33}{4} \right\}.$$

Again, take the FOC and obtain:

$$x_0 - \frac{7}{4} \cdot 2(x_0 - u_0) - \frac{9}{2} = 0$$

or

$$u_0 = \frac{9}{7} + \frac{5}{7} x_0.$$

Plugging this result back to the cost function, we obtain cost function for the firm:

$$J_0(x_0) = \frac{6}{7} x_0^2 + \frac{9}{7} x_0 + \frac{75}{14}.$$

Problem 2. *Consider the following problem of the optimal harvesting of the renewable resource. A social planner wants to maximize the present value of the utility of the representative consumer*

$$\sum_{t=0}^{\infty} \beta^t u(C_t)$$

subject to

$$Z_{t+1} = Z_t + M(1 - e^{-KZ_t}) - S_t,$$
$$Z_t \geq 0, \quad S_t \geq 0,$$

where Z_t is the current stock of renewable resource and S_t is the harvesting rate. K and M are some positive constants. There is no uncertainty and no population growth. Representative consumer receives utility from consumption C_t. Non–storable consumption good is produced using S_t as the only input:

$$C_t = Y_t = S_t^{\alpha}, \quad \alpha < 1.$$

Utility function u is twice differentiable and concave.

(a) Clearly identify state and control variables. Rewrite the problem in the following form:

$$\max_{x_t} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}),$$

where x_t is the state variable.

(b) Write Bellman equation for the problem.

(c) Derive FOC and the Envelope Theorem conditions.

(d) Substitute ET into FOC and obtain Euler equation for the problem.

(e) Derive the steady state value of stock of resources Z^* and extraction rate S^*. What happens to Z^* and S^* as representative agent becomes more (higher β) or less (lower β) "patient"?

Solution:

(a)
- **states:** Z_t,
- **controls:** C_t, S_t, Z_{t+1}, Y_t.

Using the equations for C_t we can eliminate S_t and Y_t to get

$$C_t = (Z_t - Z_{t+1} + M(1 - e^{-KZ_t}))^{\alpha}.$$

Choosing Z_t as our state, we can now rewrite our problem in the form

$$\max_{x_t} \sum_{t=0}^{\infty} \beta^t u(x_t, x_{t+1}),$$

by plugging equation for C_t into the objective. We get:

$$\sum_{t=0}^{\infty} \beta^t u(C_t(Z_t, Z_{t+1})).$$

(b) We can now rewrite the problem in the recursive form. Denote $Z_t = Z$ and $Z_{t+1} = \tilde{Z}$. Then Bellman equation is

$$V(Z) = \max_{\tilde{Z}} \left\{ u(C(Z, \tilde{Z})) + \beta V(\tilde{Z}) \right\}.$$

(c) For simplicity denote $C_i(x_1, x_2) = {}^{\partial C(x_1,x_2)}/_{\partial x_i}$, $i = 1, 2$.
FOC (derivative with respect to $\tilde{Z}$) for Bellman equation is

$$u'(C(Z, \tilde{Z})) \cdot C_2(Z, \tilde{Z}) + \beta V'(\tilde{Z}) = 0.$$

ET condition (derivative with respect to Z) is given by the equation

$$V'(Z) = u'(C(Z, \tilde{Z})) \cdot C_1(Z, \tilde{Z}).$$

(d) As usually, we shift the above equation one period ahead to get

$$V'(\tilde{Z}) = u'(C(\tilde{Z}, \tilde{\tilde{Z}})) \cdot C_1(\tilde{Z}, \tilde{\tilde{Z}}).$$

Plugging this into the FOC we can get Euler Equation

$$u'(C(Z, \tilde{Z})) \cdot C_2(Z, \tilde{Z}) + \beta u'(C(\tilde{Z}, \tilde{\tilde{Z}})) \cdot C_1(\tilde{Z}, \tilde{\tilde{Z}}) = 0.$$

Since we know that

$$C = \left(Z - \tilde{Z} + M(1 - e^{-KZ}) \right)^{\alpha},$$

we can calculate all the derivatives in EE:

$$C_1(Z, \tilde{Z}) = \alpha(1 + MKe^{-KZ}) \left(Z - \tilde{Z} + M(1 - e^{-KZ}) \right)^{\alpha - 1},$$

$$C_2(Z, \tilde{Z}) = -\alpha \left(Z - \tilde{Z} + M(1 - e^{-KZ}) \right)^{\alpha - 1},$$

and we can plug these into the equation.

(e) In steady state $Z = \tilde{Z} = \tilde{\tilde{Z}} = Z^*$. We have the following equations to determine the steady–state values:

$$\beta(1 + MKe^{-KZ^*}) = 1 \quad \Rightarrow \quad Z^* = \frac{1}{K}\ln\left(\frac{\beta MK}{1-\beta}\right),$$

$$Z^* = Z^* + M(1 - e^{-KZ^*}) - S^* \quad \Rightarrow \quad S^* = M(1 - e^{-KZ^*}) = \frac{\beta MK - 1 + \beta}{\beta K},$$

where the first equation is the Euler Equation evaluated at steady–state values of Z^*. When β increases, Z^* and S^* increase too (because ln is an increasing function).

Problem 3. *Consider a consumer with utility function*

$$\sum_{t=0}^{\infty} \beta^t u(c_t) = \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma}, \quad 0 < \beta < 1, \quad \sigma > 0.$$

The consumer is endowed with a cake of size x_0 at time $t = 0$. Each period, she has cake x_t and can either consume some, c_t, or hold some cake over to next period, x_{t+1}.

Guess that the value function $V(x_t)$ is of the following form:

$$V(x_t) = \alpha \frac{x_t^{1-\sigma}}{1-\sigma},$$

for some unknown coefficient $\alpha > 0$. Solve for the unknown coefficient. Given your value function, calculate the optimal policy. (Hint: You will need to derive both FOC and ET to solve this problem.)

Solution: The budget constraint for this problem has the following form:

$$x_{t+1} = x_t - c_t.$$

- **states:** x_t,
- **controls:** c_t, x_{t+1}.

Bellman equation is

$$V(x_t) = \max_{x_{t+1}} \{u(x_t - x_{t+1}) + \beta V(x_{t+1})\}.$$

FOC (derivative with respect to x_{t+1}) for Bellman equation is

$$-u'(x_t - x_{t+1}) + \beta V'(x_{t+1}) = 0.$$

Substituting our guess for value function, we get

$$-(x_t - x_{t+1})^{-\sigma} + \beta\alpha x_{t+1}^{-\sigma} = 0 \quad \Rightarrow \quad x_{t+1} = \frac{x_t}{1 + (\beta\alpha)^{-1/\sigma}}.$$

ET condition (derivative with respect to x_t) is given by the equation

$$V'(x_t) = u'(x_t - x_{t+1}).$$

Substituting our guess for value function, we get

$$\alpha x_t^{-\sigma} = (x_t - x_{t+1})^{-\sigma} \quad \Rightarrow \quad x_{t+1} = x_t\left(1 - \alpha^{-1/\sigma}\right).$$

Equalizing x_{t+1} derived from FOC and ET, we get

$$\frac{1}{1 + (\beta\alpha)^{-1/\sigma}} = 1 - \alpha^{-1/\sigma} \quad \Rightarrow \quad \alpha = \left(1 - \beta^{1/\sigma}\right)^{-\sigma} \quad \Rightarrow \quad x_{t+1} = \beta^{1/\sigma} x_t.$$