LNU
Spring 2017

Introduction to Dynamic Economic Models
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Exercise Session 7
Suggested Solutions

Problem 1 Consider an economy populated by N identical infinitely-lived households with his/her lifetime utility based on consuming a single good

$$\mathsf{E}_t \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\theta}}{1-\theta}$$

with $0 < \beta < 1$ and $\theta > 0$. To finance future consumption, the agent can transfer its wealth, $A_t > 0$, over time by investing into bond and equity holdings. One-period bonds earn a risk-free real gross interest rate R_t, measured in units of time $t+1$ consumption good per time t consumption good. A share of equity/tree entitles the owner to its stochastic dividend/fruit stream y_t and p_t be the share price in period t net of the period's dividend. At the beginning of period t, each tree yields nonstorable fruit or dividends in the amount y_t which is assumed to be governed by a Markov process, $y_t = s_t$, where s_t is the state of the economy. So any agent faces the following budget constraint

$$c_t + R_t^{-1} L_t + p_t M_t \leq A_t.$$

1. Write down the formula for the next period's wealth A_{t+1}.
2. Write down the household problem in this economy. What are the state and control/decision variables?
3. Write down the household's Bellman equation.
4. Derive the first-order conditions and the Euler equations. Interpret the optimality of the Euler equations economically.
5. Let us normalize the size of the economy by $N = 1$ and work with only a representative agent. Let us determine the general equilibrium of the economy, i.e. all the markets clear at the equilibrium prices. What these conditions imply for the trade in bonds and trees? What are the implications for the Euler conditions? Write down the changed Euler equations.

6. Assume now a similar economy which nevertheless differs in the following: (i) the economy is opened up so that their agents can borrow/lend to/from foreigners at the fixed gross interest rate $R > 1$, $R\beta = 1$, and (ii) all the trees are genetically modified in such sense that all of them are now identical and the delivery of the fruit is deterministic and growing according to $y_t = \lambda^t$, where $1 < \lambda < R$ so that $\sum_{t=0}^{\infty} \beta^t y_t < \infty$ holds. There is no international trade in trees. How will the Euler equations from 5 look like now?

7. Assume again the economy from 6. Derive the formula for the tree price as a function of its dividend stream?

8. Assume again the economy from 6 and assume further that the trade in the trees has prohibited.

   (a) Write down the household budget constraint and the next period wealth under this situation. Assume now additionally that there is a borrowing constraint $L_t \geq \bar{L}_t$ where $\bar{L}_t$ is either a borrowing limit given by $\bar{L}_t = 0$ in the case of the no-borrowing constraint or given by $\bar{L}_t = \tilde{L}_t$ in the case of the natural-borrowing constraint.

   (b) Explain the meaning if the natural-borrowing constraint? Write down the maximization problem of a representative household and its Lagrangian, including a borrowing constraint. What are the control/decision variables? Derive the FOC's. Derive the Euler equation. Discuss the situation when the borrowing limit is binding (nonbinding) and it's implication for the behaviour of the household.

   (c) Using the flow budget constraint above, derive the intertemporal budget constraint while imposing the transversatility condition (TVC), $\lim_{j\to\infty} R^{-j} L_{t+j} = 0$. Explain the economic meaning of the derived intertemporal budget constraint and the economic role of TVC.

   (d) Write down the maximization problem of a representative household and its Lagrangian using the intertemporal budget constraint from above instead of the flow budget constraint. What are the control/decision variables? Show that you get exactly the same FOCs as in the case of the flow budget constraint. What are the differences between the Lagrange multipliers for problem (b) and (d).

Solution: 1. The next period's wealth is given by equation

$$A_{t+1} = 1 \cdot L_t + M_t(y_{t+1} + p_{t+1}),$$

where L_t denotes bond holdings and M_t is equity holdings. The agent receives 1 from the bonds and y_{t+1} from the equity; the price of equity in the next period is p_{t+1}.

2. The household chooses consumption c_t, number of bonds L_t and equity M_t. Current wealth level A_t and y_t are taken as given (state variables). Since y_{t+1} is stochastic from

the perspective of the current period we do not consider it as neither state nor control variable. Therefore the household solves:

$$\max_{c_t, L_t, M_t} \mathsf{E} \sum_{j=0}^{\infty} \beta^j \frac{c_{t+j}^{1-\theta}}{1-\theta}$$

$$\text{s.t.} \quad c_t + R_t^{-1} L_t + p_t M_t \leq A_t, \quad A_{t+1} = L_t + M_t(s_{t+1} + p_{t+1}).$$

3. The Bellman equation is

$$V(A_t, s_t) = \max_{M_t, L_t} \{u(c_t) + \beta \mathsf{E} V(L_t + M_t(s_{t+1} + p_{t+1}), s_{t+1})\},$$

where $c_t = A_t - R_t^{-1} L_t - p_t M_t$.

4. As usual, the first order conditions are:

$$u'(c_t)(-R_t^{-1}) + \beta \mathsf{E}[V_1(A_{t+1}, s_{t+1})] = 0,$$

$$u'(c_t)(-p_t) + \beta \mathsf{E}[(s_{t+1} + p_{t+1}) V_1(A_{t+1}, s_{t+1})] = 0,$$

and the E.T. condition is:

$$V_1(A_t, s_t) = u'(c_t) \Longrightarrow V_1(A_{t+1}, s_{t+1}) = u'(c_{t+1}).$$

We can now write down the Euler equations:

$$u'(c_t) R_t^{-1} = \beta \mathsf{E}[u'(c_{t+1})],$$

$$u'(c_t) p_t = \beta \mathsf{E}[(s_{t+1} + p_{t+1}) u'(c_{t+1})].$$

5. Now assume that we have only one agent in the economy. Since the agent cannot trade with equity now, we can normalize $M_t = 1$, as he is the only one holding the "tree" in this setting. Also, we cannot use the prices p_t anymore since there is no trade in equity. Therefore the second Euler Equation no longer exists. Since the trade in bonds is not prohibited, the first Euler Equation remains the same.

6. The setting is similar to the one above. The only difference is in the condition for the gross interest rate, therefore the Euler Equation becomes:

$$u'(c_t) = \mathsf{E}[u'(c_{t+1})].$$

7. Assume again the same setting of the economy. We are interest in the price of a tree as a function of its dividend streams. The price of a tree is given by the formula

$$Q_0 = \sum_{t=0}^{\infty} \beta^t \lambda^t.$$

The conditions $R\beta = 1$ and $\lambda < R$ guarantees that $\beta\lambda < 1$, and we have

$$Q_0 = \sum_{t=0}^{\infty} \beta^t \lambda^t = \sum_{t=0}^{\infty} (\beta\lambda)^t = \frac{1}{1 - \beta\lambda}.$$

8. In this setting the agent solves

$$\max \mathsf{E}_t \sum_{t=0}^{\infty} \frac{c_{t+j}^{1-\theta}}{1-\theta}$$

$$\text{s.t.} \quad c_t + R^{-1}L_t \le A_t \ \text{ and } \ A_{t+1} = L_t + y_t.$$

Unlike in the previous setup we also have the borrowing constraint

$$L_t \ge \bar{L}_t,$$

where $\bar{L}_t$ is either 0 in case of no-borrowing limit or $\tilde{L}_t$ in case of natural-borrowing limit. We can eliminate A_t for the constraint to get

$$c_t = L_{t-1} + y_{t-1} - R^{-1}L_t.$$

The Lagrangian is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t u(c_t) + \theta_t(L_{t-1} + y_{t-1} - R^{-1}L_t - c_t) + \mu_t(L_t - \bar{L}_t)\right].$$

We have two control variables c_t and L_t, and so the FOCs are:

$$\beta^t u'(c_t) - \theta_t = 0,$$

$$-R^{-1}\theta_t + \theta_{t+1} + \mu_t = 0,$$

$$\implies u'(c_t) = u'(c_{t+1}) + \frac{\mu_t R}{\beta^t}.$$

Consider the case when borrowing constraint is non-binding. Then $\mu_t = 0$ and from the equation above we have that $c_t = c_{t+1} = \bar{c}$ is constant. Now consider the case when borrowing constraint is binding. In this case $\mu_t > 0$ and we have that $c_t < c_{t+1}$ (since $u'(c_t) > u'(c_{t+1})$ and $u'' > 0$ by assumption).

Into the equation

$$L_{t-1} = c_t + R^{-1}L_t - y_{t-1}$$

we can substitute for L_t (using the same equation but shifted one period forward). Performing the same procedure over and over again and using TVC condition we get:

$$L_{t-1} = \sum_{j=0}^{\infty} R^{-j}c_{t+j} - \sum_{j=0}^{\infty} R^{-j}y_{t+j}.$$

Setting $t = 0$ and omitting the small confusion with time subscripts we get the intertemporal budget constraint given by the equation:

$$\sum_{t=0}^{\infty} R^{-t}c_t - \sum_{t=0}^{\infty} R^{-t}y_t = 0.$$

Therefore the agent solves

$$\max \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\theta}}{1-\theta}$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} R^{-t} c_t - \sum_{t=0}^{\infty} R^{-t} y_t = 0 \ \text{ and } \ L_t \geq \bar{L}_t.$$

We can form the Lagrangian function which has the form:

$$\mathcal{L} = \sum_{t=0}^{\infty} \left[\beta^t \frac{c_t^{1-\theta}}{1-\theta} + \mu_t (L_t - \bar{L}_t) \right] - \chi \left[\sum_{t=0}^{\infty} R^{-t} c_t - \sum_{t=0}^{\infty} R^{-t} y_t \right],$$

where

$$L_t = -R c_t + R L_{t-1} + R y_t.$$

The first order condition therefore is:

$$\beta^t c^{-\theta} - \chi R^{-t} - R\mu_t = 0.$$

Assuming that the borrowing constraint is non-binding and using the condition $R\beta = 1$ we get:

$$c_t^{-\theta} = \chi$$

and

$$c_{t+1}^{-\theta} = \chi.$$

The Euler Equation can be written as:

$$\left(\frac{c_t}{c_{t+1}} \right)^{-\theta} = 1.$$

We can also compare the Lagrange multipliers with the ones we had in case of flow budged constraint:

$$\theta_t = R^{-t} \chi.$$

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