LNU
Spring 2017

Introduction to Dynamic Economic Models
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Exercise Session 6
Suggested Solutions

Problem 1 An economy consists of two infinitely lived consumers named $i = 1, 2$. There is one nonstorable consumption good. Consumer i consumes c_t^i at time t. Consumer i ranks consumption streams by

$$\sum_{t=0}^{\infty} \beta^t u(c_t^i),$$

where $\beta \in (0, 1)$ and $u(c)$ is increasing, strictly concave, and twice continuously differentiable. Consumer 1 is endowed with a stream of the consumption good $y_t^i = \{1, 0, 0, 1, 0, 0, 1, \ldots\}$. Consumer 2 is endowed with a stream of the consumption good $y_t^i = \{0, 1, 1, 0, 1, 1, 0, \ldots\}$. Assume that there are complete markets with time-0 trading.

1. Define competitive equilibrium with time 0 trading. Be careful to include definitions of all objects of which a competitive equilibrium is composed.
2. Compute a competitive equilibrium allocation with time 0 trading.
3. Suppose that one of the consumers markets a derivative asset that promises to pay 0.05 units of consumption each period. What would the price of that asset be?

Solution: *Competitive equilibrium* is an allocation of consumption sequences $\{c_t^i\}_{t=0}^{\infty}$ for consumers $i = 1, 2$ and the price system $\{q_t^0(s^t)\}_{t=0}^{\infty}$ such that

1. given the price system, the allocations solve the respective consumers maximization problems
2. markets clear
$$c_t^1 + c_t^2 = y_t^1 + y_t^2, \quad \forall t,$$

The usual first order condition for the maximization problem implies that

$$q_t^0 = \frac{\beta^t u'(c_t^i)}{u'(c_0^i)}.$$

Since utility function is increasing and strictly concave we can make a guess that consumption of each individual is equal across time and is $\bar{c}^i$. In this case

$$q_t^0 = \beta^t.$$

The budget constraint for consumer i then becomes

$$(1-\beta)\sum_{t=0}^{\infty} \beta^t y_t^i = \bar{c}^i \Longrightarrow \bar{c}^1 = (1-\beta)(1+\beta^3+\beta^6+\dots) = \frac{1}{1+\beta+\beta^2}.$$

Using market clearing condition we get that

$$\bar{c}^2 = 1 - \bar{c}^1.$$

Now, since markets are complete markets, the derivative security is redundant: it can be priced using the Arrow-Debreu price we got above. Let p_0 denote the time zero price of derivative:

$$p_0 = \sum_{t=0}^{\infty} 0.05\, q_t^0 = \frac{0.05}{1-\beta}.$$

□

Problem 2 A pure endowment economy consists of two types of consumers. Consumers of type 1 order consumption streams of the one good according to

$$\sum_{t=0}^{\infty} \beta^t \log\left(c_t^1\right)$$

and consumers of type 2 order consumption streams according to

$$\sum_{t=0}^{\infty} \beta^t c_t^2$$

where $c_t^i \geq 0$ is the consumption of a type i consumer and $\beta \in (0,1)$ is a common discount factor. The consumption good is tradable but nonstorable. There are equal numbers of the two types of consumer. The consumer of type 1 is endowed with the consumption sequence

$$y_t^1 = \begin{cases} 0 & \text{if } t = 0, 3, 6, \dots \\ \alpha & \text{if } t = 1, 4, 7, \dots \\ \mu & \text{if } t = 2, 5, 8, \dots \end{cases}$$

where $\alpha = \mu\left(1 + \frac{1}{\beta}\right)$ and $\mu > 0$. The consumer of type 2 is endowed with the consumption sequence

$$y_t^2 = \mu > 0.$$

1. Define competitive equilibrium with time 0 trading. Be careful to include definitions of all objects of which a competitive equilibrium is composed.
2. Compute a competitive equilibrium allocation with time 0 trading.
3. Compute the time t wealths of the two types of consumers using the competitive equilibrium prices.

Solution: *Competitive equilibrium* is an allocation of consumption sequences $\{c_t^1, c_t^2\}_{t=0}^{\infty}$ and the price system $\{q_t^0\}_{t=0}^{\infty}$ such that, given the price system

1. the allocation $\{c_t^1\}_{t=0}^{\infty}$ solves the maximization problem of consumer 1

$$\max \sum_{t=0}^{\infty} \beta^t \log(c_t^1),$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} q_t^0 c_t^1 \leq \sum_{t=0}^{\infty} q_t^0 y_t^1,$$

2. the allocation $\{c_t^2\}_{t=0}^{\infty}$ solves the maximization problem of consumer 2

$$\max \sum_{t=0}^{\infty} \beta^t c_t^2,$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} q_t^0 c_t^2 \leq \sum_{t=0}^{\infty} q_t^0 y_t^2,$$

3. and markets clear

$$c_t^1 + c_t^2 = y_t^1 + y_t^2, \quad \forall t.$$

Now we can solve for CE allocation. Lagrangian function for consumer 1 is given by

$$\mathcal{L}^1 = \sum_{t=0}^{\infty} [\beta^t \log(c_t^1) - \lambda^1 q_t^0 (c_t^1 - y_t^1)].$$

F.O.C. is given by equation

$$\beta^t \frac{1}{c_t^1} = \lambda^1 q_t^0.$$

Since consumer 2 has linear utility, we need to **assume interior solution**. In this case F.O.C. becomes:

$$\beta^t = \lambda^2 q_t^0 \Longrightarrow \lambda_2 = 1 \text{ when } q_0^0 = 1 \Longrightarrow q_t^0 = \beta^t.$$

This implies that for consumer 1 we have:

$$c_t^1 = \frac{1}{\lambda^1},$$

meaning that his consumption is constant over time: $c_t^1 = c_{t+1}^1 = \bar{c}^1$. Therefore, the budget constraint for consumer 1 becomes:

$$\sum_{t=0}^{\infty} \beta^t \bar{c}^1 = \sum_{t=0}^{\infty} \beta^t y_t^1 = \sum_{t=0}^{\infty} 0 \cdot \beta^{3t} + \sum_{t=0}^{\infty} \alpha \beta^{3t+1} + \sum_{t=0}^{\infty} \mu \beta^{3t+2} \Longrightarrow$$

$$\Longrightarrow \bar{c}^1 = \frac{\alpha\beta + \mu\beta^2}{1 + \beta + \beta^2} = \mu.$$

Since consumer 1 wants to enjoy smooth consumption over time, he has to borrow from consumer 2 at some periods. Consumer 2 is risk-neutral, so he will be willing to lend to consumer 1. Since consumer 1 receives different endowments every third period, we have three cases now:

- Let $t = 3\tau$. Then market clearing condition implies

$$\bar{c}^1 + c_t^2 = \mu + 0 \Longrightarrow c_t^2 = \mu - \mu = 0.$$

- Let $t = 3\tau + 1$. Then market clearing condition implies

$$\bar{c}^1 + c_t^2 = \mu + \alpha \Longrightarrow c_t^2 = \mu + \alpha - \mu = \alpha.$$

- Let $t = 3\tau + 2$. Then market clearing condition implies

$$\bar{c}^1 + c_t^2 = \mu + \mu \Longrightarrow c_t^2 = \mu + \mu - \mu = \mu.$$

Now we need to compute the time t wealth. It is given by the formula:

$$W_t^1 = \sum_{\tau=i}^{\infty} q_\tau^0 (y_\tau^1 - c_\tau^1), \quad W_t^2 = \sum_{\tau=t}^{\infty} q_\tau^0 (y_\tau^2 - c_\tau^2)$$

for consumers 1 and 2 respectively. Again, we have three cases depending at what period we start. Consider the case when $t = 3i$ (with other cases left as an exercise). For consumer 1 we have:

$$W_t^1 = \sum_{\tau=i}^{\infty} \left(0 \cdot \beta^{3\tau} + \alpha \cdot \beta^{3\tau+1} + \mu \cdot \beta^{3\tau+2}\right) - \mu \sum_{\tau=i}^{\infty} \beta^{3\tau} =$$

$$= (\alpha\beta + \mu\beta^2) \sum_{\tau=i}^{\infty} \beta^{3\tau} - \mu \sum_{\tau=i}^{\infty} \beta^{3\tau} =$$

$$= (\alpha\beta + \mu\beta^2) \frac{\beta^{3i}}{1 - \beta^3} - \mu \frac{\beta^{3i}}{1 - \beta^3}.$$

In order to calculate the wealths when $t = 3i + 1$ and $t = 3i + 2$ note that for consumer 1:

$$\text{for } t = 3i + 1, \quad W_{3i+1}^1 = W_{3i}^1 - q_\tau^0 (y_\tau^1 - c_\tau^1) \Big|_{\tau=3i} =$$

$$= (\alpha\beta + \mu\beta^2) \frac{\beta^{3i}}{1 - \beta^3} - \mu \frac{\beta^{3i}}{1 - \beta^3} - \beta^{3i}(\alpha - \mu),$$

and

$$\text{for } t = 3i+2, \quad W^1_{3i+2} = W^1_{3i} - q^0_\tau (y^1_\tau - c^1_\tau)\Big|_{\tau=3i} - q^0_\tau (y^1_\tau - c^1_\tau)\Big|_{\tau=3i+1} =$$

$$= (\alpha\beta + \mu\beta^2)\frac{\beta^{3i}}{1-\beta^3} - \mu\frac{\beta^{3i}}{1-\beta^3} - \beta^{3i}(\alpha - \mu) - \beta^{3i+1}(\alpha + \mu - \mu),$$

Since the market clears in every period, the following equalities hold

$$\text{for every } t \quad W^2_t = -W^1_t.$$

□