LNU
Spring 2017

Introduction to Dynamic Economic Models
Instructor: Mykola Babiak
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Exercise Session 2 (Miscellaneous)
Suggested Solutions

Problem 1 (Guess-and-Verify Method: Lucky Guess) Consider the problem of consumer who seeks to solve

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\log c_t + \gamma \log c_{t-1}), \;\; 0 < \gamma < 1$$

subject to the following constraints:

$$k_{t+1} + c_t \leq A k_t^{\alpha}, \; A > 0, \; \alpha \in (0,1),$$
$$c_t > 0, \; k_t > 0, \; k_0, c_{-1} \text{ given.}$$

1. Clearly identify state and control variable(s). Set up the Bellman equation for the problem (that is write the problem in the recursive form). *Hint:* Value function for the consumer will be the function of two variables.
2. Guessing that the value function is of the form:
$$V(k_t, c_{t-1}) = E + F \log k_t + G \log c_{t-1},$$
derive constants E, F and G. Calculate the optimal policy given your value function.

Solution: We start by identifying state and control variables:

- **states:** k_t, c_{t-1}
- **controls:** k_{t+1}, c_t

We eliminate one control using the equation

$$c_t = A k_t^{\alpha} - k_{t+1},$$

and the Bellman equation becomes

$$V(k, c_-) = \max_{\tilde{k}} \left\{ \log(A k^{\alpha} - \tilde{k}) + \gamma \log(c_-) + \beta V(\tilde{k}, A k^{\alpha} - \tilde{k}) \right\},$$

where $k = k_t,\ \tilde{k} = k_{t+1},\ c_- = c_{t-1}$. We make a guess that

$$V(k, c_-) = E + F\log(k) + G\log(c_-).$$

Now substituting for V into the Bellman equation we get

$$E+F\log(k)+G\log(c_-) = \max_{\tilde{k}}\left\{\log(Ak^\alpha - \tilde{k}) + \gamma\log(c_-) + \beta\left(E + F\log(\tilde{k}) + G\log(Ak^\alpha - \tilde{k})\right)\right\}.$$

The F.O.C. (derivative with respect to $\tilde{k}$) is:

$$-\frac{1}{Ak^\alpha - \tilde{k}} + \beta F\frac{1}{\tilde{k}} - \beta G\frac{1}{Ak^\alpha - \tilde{k}} = 0 \Longrightarrow \tilde{k} = \frac{\beta F Ak^\alpha}{\beta(F+G)+1}.$$

E.T. condition with respect to c_- is

$$\frac{G}{c_-} = \frac{\gamma}{c_-} \Longrightarrow G = \gamma.$$

E.T. condition with respect to k is

$$\frac{F}{k} = \frac{\alpha Ak^{\alpha-1}}{Ak^\alpha - \tilde{k}} + \beta G\frac{\alpha Ak^{\alpha-1}}{Ak^\alpha - \tilde{k}},$$

$$\frac{F}{k} = \frac{\alpha Ak^{\alpha-1}}{Ak^\alpha - \frac{\beta F Ak^\alpha}{\beta(F+G)+1}} + \beta G\frac{\alpha Ak^{\alpha-1}}{Ak^\alpha - \frac{\beta F Ak^\alpha}{\beta(F+G)+1}},$$

$$\frac{F}{k} = \frac{\alpha Ak^{\alpha-1}}{Ak^\alpha - \frac{\beta F Ak^\alpha}{\beta(F+\gamma)+1}} + \beta\gamma\frac{\alpha Ak^{\alpha-1}}{Ak^\alpha - \frac{\beta F Ak^\alpha}{\beta(F+\gamma)+1}} \Longrightarrow F = \frac{\alpha(1+\beta\gamma)}{1-\alpha\beta}.$$

Substituting for G, F and $\tilde{k}$ into the Bellman equation and solving for E (omitting tedious algebra here) we get:

$$E = \frac{1}{1-\beta}\left(\frac{1+\beta\gamma}{1-\alpha\beta}\log\alpha\beta A - (1-\beta\gamma)\log\frac{\alpha\beta}{1-\alpha\beta}\right).$$

Also we can solve for $\tilde{k}$

$$\tilde{k} = \alpha\beta Ak^\alpha.$$

□

Problem 2 (Guess-and-Verify Method: Failure) Consider the household that seeks to maximize his lifetime utility

$$\max_{\{c_t\}_{t=0}^\infty}\sum_{t=0}^{\infty}\beta^t\,\frac{c_t^{1-\sigma}-1}{1-\sigma},$$

subject to the following constraint:

$$k_{t+1} + c_t = Ak_t^\alpha + (1-\delta)k_t,\ \ 0<\delta<1.$$

Show that guessing that policy function is

- constant, that is $V(k_t) = C$,
- linear in state (excluding constant), that is $V(k_t) = Ck_t$

does not work.

Solution: First assume that our guess is $V(k_t) = C$, where C is some constant. Then, the Bellman equation is

$$V(k_t) = \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{1-\sigma} - 1}{1-\sigma} + \beta V(k_{t+1}) \right\}.$$

Taking the F.O.C. (the derivative with respect to $\tilde{k}$) we get:

$$(-1)\left(Ak_t^\alpha + (1-\delta) - k_{t+1}^{-\sigma}\right) = 0,$$

which has unique solution when $c = 0$. However, this is not optimal, and thus we conclude that our guess was wrong.

Now assume that our guess is that $V(k_t) = Ck_t$. Again, the Bellman equation is

$$V(k_t) = \max_{k_{t+1}} \left\{ \frac{(Ak_t^\alpha + (1-\delta)k_t - k_{t+1})^{1-\sigma} - 1}{1-\sigma} + \beta V(k_{t+1}) \right\}.$$

Taking the F.O.C. and substituting for V we get

$$(-1)(Ak_t^\alpha + (1-\delta) - k_{t+1})^{-\sigma} + C\beta = 0 \quad \Longrightarrow \quad k_{t+1} = Ak_t^\alpha + k_t(1-\delta) - (\beta C)^{-\frac{1}{\sigma}}.$$

Plugging this back the Bellman equation, and equating terms next to k^α we get that

$$\beta AC = 0,$$

which is not possible since $\beta \in (0,1)$ and $A > 0$. Again, our guess was wrong. □

Problem 3 (Value Function Iteration: Analytical Solution) A planner chooses a sequence $\{c_t, k_{t+1}\}_{t=0}^\infty$ to maximize

$$\sum_{t=0}^{\infty} \beta^t \ln(c_t)$$

subject to a given value for k_0 and a transition law

$$k_{t+1} + c_t = Ak_t^\alpha,$$

where $A > 0$, $\alpha \in (0,1)$, $\beta \in (0,1)$. Solve this problem using value function iterations.

Solution:

- iteration 1. Let's start with $V_0(x) = 0$. The problem that we want to solve is

$$V_1(k) = \max_{\widetilde{k},c} \left\{ \ln c + \beta V_0(\widetilde{k}) \right\}$$
$$\text{s.t.} \quad c + \widetilde{k} = Ak^\alpha$$

or

$$V_1(k) = \max_{\widetilde{k}} \left\{ \ln(Ak^\alpha - \widetilde{k}) + \beta V_0(\widetilde{k}) \right\} = \max_{\widetilde{k}} \left\{ \ln(Ak^\alpha - \widetilde{k}) \right\}$$

since $\ln(\cdot)$ is increasing function, maximum is reached when $\widetilde{k} = 0$ and thus $c = Ak^\alpha$ and $V_1(k) = \ln(Ak^\alpha) = \ln A + \alpha \ln k$.

- iteration 2. We plug $V_1(k)$ into the Bellman equation, so the problem that we are solving now is

$$V_2(k) = \max_{\widetilde{k},c} \left\{ \ln c + \beta V_1(\widetilde{k}) \right\}$$
$$\text{s.t.} \quad c + \widetilde{k} = Ak^\alpha$$

or

$$V_2(k) = \max_{\widetilde{k}} \left\{ \ln(Ak^\alpha - \widetilde{k}) + \beta V_1(\widetilde{k}) \right\} = \max_{\widetilde{k}} \left\{ \ln(Ak^\alpha - \widetilde{k}) + \beta(\ln A + \alpha \ln \widetilde{k}) \right\}$$

the FOC is

$$[\widetilde{k}] : \qquad \frac{\partial}{\partial \widetilde{k}} \left\{ \ln(Ak^\alpha - \widetilde{k}) + \beta(\ln A + \alpha \ln \widetilde{k}) \right\} \stackrel{!}{=} 0$$

so we have

$$-\frac{1}{Ak^\alpha - \widetilde{k}} + \alpha\beta \frac{1}{\widetilde{k}} = 0$$
$$\widetilde{k} = \alpha\beta(Ak^\alpha - \widetilde{k})$$

thus we have as the updated optimal policies

$$\widetilde{k}(k) = \frac{\alpha\beta A k^\alpha}{1 + \alpha\beta} \qquad c(k) = \frac{Ak^\alpha}{1 + \alpha\beta}$$

and the updated approximation of value function

$$\begin{aligned} V_2(k) &= \ln\left(\frac{Ak^\alpha}{1+\alpha\beta}\right) + \beta V_1\left(\frac{\alpha\beta A k^\alpha}{1+\alpha\beta}\right) = \ln\left(\frac{Ak^\alpha}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A k^\alpha}{1+\alpha\beta}\right) \\ &= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln k \end{aligned}$$

- iteration 3. We plug $V_2(k)$ into the Bellman equation, the problem that we are solving is

$$V_3(k) = \max_{\widetilde{k},c} \left\{ \ln c + \beta V_2(\widetilde{k}) \right\}$$
$$\text{s.t.} \quad c + \widetilde{k} = Ak^{\alpha}$$

or

$$\begin{aligned} V_3(k) &= \max_{\widetilde{k}} \left\{ \ln(Ak^{\alpha} - \widetilde{k}) + \beta V_2(\widetilde{k}) \right\} \\ &= \max_{\widetilde{k}} \left\{ \ln(Ak^{\alpha} - \widetilde{k}) + \beta \left[\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln \widetilde{k} \right] \right\} \end{aligned}$$

the FOC is

$$[\widetilde{k}] : \frac{\partial}{\partial \widetilde{k}} \left\{ \ln(Ak^{\alpha} - \widetilde{k}) + \beta \left[\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln \widetilde{k} \right] \right\} \stackrel{!}{=} 0$$

so we have

$$-\frac{1}{Ak^{\alpha} - \widetilde{k}} + \alpha\beta \frac{1+\alpha\beta}{\widetilde{k}} = 0$$
$$\widetilde{k} = (\alpha\beta + \alpha^2\beta^2)(Ak^{\alpha} - \widetilde{k})$$

thus we have as the optimal policies

$$\widetilde{k}(k) = \frac{(\alpha\beta + \alpha^2\beta^2)Ak^{\alpha}}{1+\alpha\beta+\alpha^2\beta^2} \qquad c(k) = \frac{Ak^{\alpha}}{1+\alpha\beta+\alpha^2\beta^2}$$

and the updated approximation of value function

$$\begin{aligned} V_3(k) &= \beta^2 \ln A + \beta \ln \frac{A}{1+\alpha\beta} + \ln \frac{A}{1+\alpha\beta+\alpha^2\beta^2} + \\ &\quad + \beta\alpha\beta \ln \frac{\alpha\beta}{1+\alpha\beta} A + (\alpha\beta + \alpha^2\beta^2) \ln \frac{\alpha\beta+\alpha^2\beta^2}{1+\alpha\beta+\alpha^2\beta^2} A + \\ &\quad + \alpha(1+\alpha\beta+\alpha^2\beta^2)\ln k \end{aligned}$$

- we can proceed further with fourth iteration, where we will find that the optimal policies are

$$\widetilde{k}(k) = \frac{(\alpha\beta + \alpha^2\beta^2 + \alpha^3\beta^3)Ak^{\alpha}}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3} \qquad c(k) = \frac{Ak^{\alpha}}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3}$$

and value function is

$$\begin{aligned} V_4(k) &= \beta^3 \ln A + \beta^2 \ln \frac{A}{1+\alpha\beta} + \beta \ln \frac{A}{1+\alpha\beta+\alpha^2\beta^2} + \ln \frac{A}{1+\alpha\beta+\ldots+\alpha^3\beta^3} \\ &\quad + \beta^2\alpha\beta \ln \frac{\alpha\beta}{1+\alpha\beta} A + \beta(\alpha\beta+\alpha^2\beta^2) \ln \frac{\alpha\beta+\alpha^2\beta^2}{1+\alpha\beta+\alpha^2\beta^2} A \\ &\qquad + (\alpha\beta+\ldots+\alpha^3\beta^3) \ln \frac{\alpha\beta+\ldots+\alpha^3\beta^3}{1+\alpha\beta+\ldots+\alpha^3\beta^3} A \\ &\quad + \alpha(1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3)\ln k \end{aligned}$$

- in general after iteration n we therefore have the optimal policies

$$\widetilde{k}(k) = \frac{(\alpha\beta + \alpha^2\beta^2 + \ldots + \alpha^{n-1}\beta^{n-1})Ak^\alpha}{1 + \alpha\beta + \alpha^2\beta^2 + \cdots + \alpha^{n-1}\beta^{n-1}} \qquad c(k) = \frac{Ak^\alpha}{1 + \alpha\beta + \alpha^2\beta^2 + \ldots + \alpha^{n-1}\beta^{n-1}}$$

and value function

$$\begin{aligned}
V_n(k) &= \sum_{t=0}^{n-1} \beta^t \ln \frac{A}{\sum_{i=0}^{n-1-t}(\alpha\beta)^i} + \sum_{t=0}^{n-2}\left[\beta^t\alpha\beta \sum_{i=0}^{n-2-t}(\alpha\beta)^i \ln \frac{\alpha\beta A \sum_{i=0}^{n-2-t}(\alpha\beta)^i}{\sum_{i=0}^{n-1-t}(\alpha\beta)^i}\right] + \\
&\quad + \alpha(1 + \alpha\beta + \alpha^2\beta^2 + \ldots + \alpha^{n-1}\beta^{n-1}) \ln k \\
&= \sum_{t=0}^{n-1} \beta^t \ln \frac{A}{\frac{1-(\alpha\beta)^{n-t}}{1-\alpha\beta}} + \sum_{t=0}^{n-2}\left[\beta^t\alpha\beta \frac{1-(\alpha\beta)}{1-\alpha\beta}^{n-1-t} \ln \frac{\frac{1-(\alpha\beta)}{1-\alpha\beta}^{n-1-t}}{\frac{1-(\alpha\beta)}{1-\alpha\beta}^{n-t}} \alpha\beta A\right] + \\
&\quad + \alpha(1 + \alpha\beta + \alpha^2\beta^2 + \ldots + \alpha^{n-1}\beta^{n-1}) \ln k
\end{aligned}$$

taking a limit for $n \to \infty$ we have the optimal policies

$$\lim_{n\to\infty} \widetilde{k}_n(k) = \lim_{n\to\infty} \frac{(\alpha\beta + \alpha^2\beta^2 + \ldots + \alpha^{n-1}\beta^{n-1})Ak^\alpha}{1 + \alpha\beta + \alpha^2\beta^2 + \cdots + \alpha^{n-1}\beta^{n-1}} = \frac{\sum_{i=0}^{\infty}(\alpha\beta)^i}{\sum_{i=0}^{\infty}(\alpha\beta)^i}\alpha\beta A k^\alpha = \alpha\beta A k^\alpha$$

$$\lim_{n\to\infty} c_n(k) = \lim_{n\to\infty} (Ak^\alpha - \widetilde{k}_n(k)) = (1 - \alpha\beta A k^\alpha)$$

and the value function

$$\begin{aligned}
\lim_{n\to\infty} V_n(k) &= \lim_{n\to\infty} \Bigg\{ \sum_{t=0}^{n-1} \beta^t \ln \frac{A}{\frac{1-(\alpha\beta)}{1-\alpha\beta}^{n-t}} + \sum_{t=0}^{n-2}\left[\beta^t\alpha\beta \frac{1-(\alpha\beta)}{1-\alpha\beta}^{n-1-t} \ln \frac{\frac{1-(\alpha\beta)}{1-\alpha\beta}^{n-1-t}}{\frac{1-(\alpha\beta)}{1-\alpha\beta}^{n-t}} \alpha\beta A\right] \\
&\quad + \alpha(1 + \alpha\beta + \alpha^2\beta^2 + \ldots + \alpha^{n-1}\beta^{n-1}) \ln k \Bigg\} \\
&= \sum_{t=0}^{\infty} \beta^t \ln \frac{A}{\frac{1}{1-\alpha\beta}} + \sum_{t=0}^{\infty}\left[\beta^t\alpha\beta\frac{1}{1-\alpha\beta} \ln \frac{\frac{1}{1-\alpha\beta}}{\frac{1}{1-\alpha\beta}}\alpha\beta A\right] + \alpha \sum_{t=0}^{\infty}(\alpha\beta)^t \ln k \\
&= \frac{1}{1-\beta}\ln(1-\alpha\beta)A + \frac{1}{1-\beta}\frac{\alpha\beta}{1-\alpha\beta}\ln \alpha\beta A + \frac{\alpha}{1-\alpha\beta}\ln k
\end{aligned}$$

$\square$

Extra: Analytical dynamic programming without substitution We have seen how we can derive the first order conditions, envelope conditions and euler equation by substituting out one variable by use of the constraint. This approach yields the desired results and is usefull especially in some situtiations. However, this requires that we substitute out the 'correct' variable (for example, try to substitute out k_{t+1} instead of c_t, to see where it leads). More importatly, sometimes we cannot simplify the constraint to get closed form solution for the variables we want to substitute out. For these situations, and also for the basic situations, we can decide not to substitute out and leave the constraint in the eqaution. This note is going to illustrate this approach, which might be useful to know in some situations.

Consider the basic Robinson Crusoe problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

$$\text{s.t. } c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$$

$$k_0 \quad \text{given}$$

Normally we would substitute consumption out and write the Bellan equation as follows:

$$V(k) = \max_{\tilde{k}} \left\{ u\left[f(k) + (1-\delta)k - \tilde{k} \right] + \beta V(\tilde{k}) \right\}$$

and proceed with FOCs and ET.

However, we can add the constraint at the end of the standard Bellman equation with lagrange multiplier as we would in situation when we would not set up the problem in terms of dynamic programming:

$$V(k) = \max_{c,\tilde{k}} \left\{ u(c) + \beta V(\tilde{k}) - \lambda_t(c + \tilde{k} - f(k) + (1-\delta)k) \right\}$$

So the relationship between choice and state variables is fully captured through the langrangian and lagrange multiplier becomes new variable in the problem. Note that the signs matter as in usual problem (i.e. $+\lambda$ and $-\lambda$ yiled two different problems). Its enough to remember that the shadow value of current capital (i.e. our budget) has to be positive and hence the sign of lagrange multiplier and sign of current capital k_t must together give positive sign. We could easily put $+\lambda$ and change the sings inside of the bracket. Also note that the lagrange multiplier is time-varysing!

Now we want illustrate the solution method by solving the model. Derive the FOC as normally, taking into account the constraint:

$$\text{F.O.C. w.r.t. c:} \qquad \frac{\partial V(k)}{\partial c} = u'(c) - \lambda_t = 0$$

Note that since all the relationship between our variables are captured in the multiplier, we do not consider the usual indirect effects. This however means that we have to take FOC w.r.t. the other choice as well:

$$\text{F.O.C. w.r.t. } \tilde{k}: \qquad \frac{\partial V(k)}{\partial k} = \beta \frac{\partial V(\tilde{k})}{\partial \tilde{k}} - \lambda_t = 0$$

Now we can substitute out the lagrange multiplier from the firts equation into second equation to obtain:

$$\beta \frac{\partial V(\tilde{k})}{\partial \tilde{k}} = u'(c)$$

To eliminate the partial of the value function we use the evelope theorem as usual:

$$\text{E.T.:} \qquad \frac{\partial V(k)}{\partial k} = \lambda_t \left(f'(k) + 1 - \delta\right)$$

First, again substitute out the multiplier from first equation:[1]

$$\frac{\partial V(k)}{\partial k} = u'(c) \left(f'(k) + 1 - \delta\right)$$

And finally substitute after shifting forward back in the FOC for k :

$$u'(c) = \beta u'(\tilde{c}) \left(f'(\tilde{k}) + 1 - \delta\right)$$

This is the standard EE equation obtained in the substitution way. Note that we could have substituted out $u'(c)$ instead and obtain the EE in the (unobserved) lagrange multiplier, which would say exactly the same thing in different way. In some problems this is easier than trying to substitute the multiplier out.

I personally prefer to use this method in all problems because I think its less prone to mistakes, as it requires less manipulation. However, the great advantage is in situation when substitution is complicated and hence this approach simplifies our life. Furthermore, by eliminating indirect relationships between variables it is less prone to the typical question 'should I take derivative with respect to this variables as well?'.

[1] Alternatively you could shift forward and then substitute into following equation:

$$\lambda_{t+1} \left(f'(\tilde{k}) + 1 - \delta\right) \quad \text{with} \quad \lambda_{t+1} = u'(\tilde{c}) \quad \text{from FOC}$$