LNU
Spring 2017

Introduction to Dynamic Economic Models
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Exercise Session 1
Suggested Solutions

Problem 1 (Bellman Equation: Basic) Consider social planner's problem of maximizing lifetime utility of the representative consumer:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the following constraints:

$$\begin{aligned} y_t &= c_t + i_t, \\ y_t &= f(k_t), \\ k_{t+1} &= (1-\delta)k_t + i_t, \\ k_{t+1} &\geq 0,\ c_t \geq 0, \\ k_0 &\text{ is given,} \end{aligned}$$

where c is consumption, i is investment, k is capital and output y is produced from capital using production function $f(\cdot)$; δ is the depreciation rate of capital.

1. Clearly identify state and control variables. Set up the Bellman equation for the problem (that is write the problem in the recursive form).
2. Derive First Order Conditions and Envelope Theorem conditions.
3. Using the equations from above derive Euler equation.
4. Derive expression for steady state level of capital and consumption in terms of parameters of the model. You can assume that $f(k) = k^\alpha$ for $\alpha < 1$.

Solution: 1. There is only single state, k_t, but many controls. All of c_t (consumption in current period), i_t (investment in current period) and k_{t+1} (capital in future period) are control variables. However, these three are interlinked so not all of them can be chosen at once. To see this lets first simplify the constraints:

$$c_t = y_t - i_t \quad \ldots\text{from first constraint}$$

$$i_t = k_{t+1} - (1-\delta)k_t \quad \ldots \text{from third constraint}$$

Combining both with second constraint yields:

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1}$$

Therefore, choosing k_{t+1} determines c_t and vice versa. In conclusion, we identify state and control variables as follows:

- states: k_t,
- controls: c_t and k_{t+1}.

2. Using the constraint we can rewrite objective function as

$$\sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1-\delta)k_t - k_{t+1})$$

Thus we are left with one state variable k_t and one control variable k_{t+1}. Denote $k_t = k$ and $k_{t+1} = \tilde{k}$. Now we can set up the Bellman equation for our problem:

$$V(k) = \max_{\tilde{k}} \left\{ u(f(k) + (1-\delta)k - \tilde{k}) + \beta V(\tilde{k}) \right\}.$$

We take the F.O.C. with respect to $\tilde{k}$ and get:

$$-u'\left(f(k) + (1-\delta)k - \tilde{k}\right) + \beta V'(\tilde{k}) = 0.$$

3. E.T. condition (we get by differentiating Bellman equation w.r.t. k) is:[1]

$$V'(k) = u'\left(f(k) + (1-\delta)k - \tilde{k}\right)((1-\delta) + f'(k))$$

Now we shift E.T. condition one period ahead to get:

$$V'(\tilde{k}) = u'\left(f(\tilde{k}) + (1-\delta)\tilde{k} - \tilde{\tilde{k}}\right)((1-\delta) + f'(\tilde{k})).$$

[1] For illustration I will derive it explicitly here:

$$\frac{\partial V(k)}{\partial k} = \frac{\partial u\big(c(k,\tilde{k})\big)}{\partial c}\frac{\partial c}{\partial k} + \frac{\partial u\big(c(k,\tilde{k})\big)}{\partial c}\frac{\partial c}{\partial \tilde{k}}\frac{\partial \tilde{k}}{\partial k} + \beta\frac{\partial V\big(\tilde{k}(k)\big)}{\partial \tilde{k}}\frac{\partial \tilde{k}}{\partial k}$$

$$\frac{\partial V(k)}{\partial k} = u'(c)((1-\delta) + f'(k)) + u'(c)(-1)\frac{\partial \tilde{k}}{\partial k} + \beta\frac{\partial V\big(\tilde{k}(k)\big)}{\partial \tilde{k}}\frac{\partial \tilde{k}}{\partial k}$$

The crucial thing is to realize that the terms that include $\frac{\partial \tilde{k}}{\partial k}$ can be collected to yield FOC:

$$\frac{\partial V(k)}{\partial k} = u'(c)((1-\delta) + f'(k)) + \left\{ \overbrace{u'(c)(-1) + \beta\frac{\partial V\big(\tilde{k}(k)\big)}{\partial \tilde{k}}}^{=0 \text{ by FOC}} \right\} \frac{\partial \tilde{k}}{\partial k}$$

Thus the last term drops out, yielding our envelope theorem condition.

We plug the last equation to F.O.C. and get:

$$-u'(f(k)+(1-\delta)k-\tilde{k})+\beta u'(f(\tilde{k})+(1-\delta)\tilde{k}-\tilde{\tilde{k}})((1-\delta)+f'(\tilde{k}))=0.$$

Rearranging, we can get Euler Equation:

$$\frac{u'(f(k)+(1-\delta)k-\tilde{k})}{u'(f(\tilde{k})+(1-\delta)\tilde{k}-\tilde{\tilde{k}})}=\beta(f'(\tilde{k})+(1-\delta))$$

Substituting back for c and rearanging we get the standard form:

$$u'(c')=\beta u'(\tilde{c})\left(f'(\tilde{k})+(1-\delta)\right)$$

4. Now to derive the steady state values of capital and consuption in terms of the parameters (β,δ) of the model we have to set $c=\tilde{c}=c_{ss}$ and $k=\tilde{k}=k_{ss}$ and plug them into Euler equation and the budget constraint:

$$u'(c_{ss})=\beta u'(c_{ss})\left(f'(k_{ss})+(1-\delta)\right)$$

Since the arguments of marginal utility on both sides are the same, we can cancel the terms to obtain:

$$1=\beta(f'(k_{ss})+(1-\delta))$$
$$\frac{1}{\beta}-(1-\delta)=f'(k_{ss})$$
$$k_{ss}=f'^{-1}\left(\frac{1}{\beta}-(1-\delta)\right)$$

For consuption we use th transiton law:

$$c_{ss}=f(k_{ss})+(1-\delta)k_{ss}-k_{ss}$$

Now simply substitute k_{ss} out and get final result.

$\square$

Problem 2 (Bellman Equation: Multiple States) Consider the problem of consumer who seeks to solve

$$\max_{\{c_t\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t(\log c_t+\gamma\log c_{t-1})\qquad\text{with}\quad 0<\gamma<1$$

subject to the following constraints:

$$k_{t+1}+c_t\leq Ak_t^{\alpha}\quad A>0,\quad\alpha\in(0,1)$$

$$c_t\leq 0,\quad k_t>0,\quad k_0,c_{-1}\text{ given}$$

1. Clearly identify state and control variable(s).
2. Set up the Bellman equation for the problem (that is write the problem in the recursive form). *Hint:* Value function for the consumer will be the function of two variables.
3. Derive FOC, ET and EE.
4. Derive expression for steady state level of capital and consuption in terms of parameters of the model.

Solution: 1. The state and control variables are:

- states: k_t, c_{-1}
- controls: c_t, i_t, k_{t+1}.

2. The Bellman function has follwoing from:

$$V(k, c_{-1}) = \max_{\tilde{k}, c} \left\{ u(c) + \beta V(\tilde{k}, c) \right\} \qquad \text{s.t} \qquad c = Ak^\alpha - \tilde{k}$$

or after substitution:

$$V(k, c_{-1}) = \max_{\tilde{k}} \left\{ \log(Ak^\alpha - \tilde{k}) + \gamma \log c_{-1} + \beta V(\tilde{k}, c) \right\}.$$

3. We proceed as usually:

$$\text{FOC:} \qquad (-1)\frac{1}{c} + \beta \left[V_1(\tilde{k}, c) + V_2(\tilde{k}, c) \frac{\partial c}{\partial \tilde{k}} \right] = 0$$

$$\text{ET w.r.t. } k: \qquad \frac{\partial V(k, c_{-1})}{\partial k} = \frac{1}{Ak^\alpha - \tilde{k}} (\alpha A k^{\alpha - 1}) + \beta \frac{\partial V(\tilde{k}, c)}{\partial c} (\alpha A k^{\alpha - 1})$$

$$\text{ET w.r.t. } c_{-1}: \qquad \frac{\partial V(k, c_{-1})}{\partial c_{-1}} = \gamma \frac{1}{c_{-1}}$$

Since we have future value function w.r.t. c in the first envelope condition, we have to substitute it there (after forwarding it one period) before substituting that envelope condition into FOC:

$$\frac{\partial V(k, c_{-1})}{\partial k} = \frac{1}{Ak^\alpha - \tilde{k}} (\alpha A k^{\alpha - 1}) + \beta \gamma \frac{1}{c} (\alpha A k^{\alpha - 1}).$$

Substituting both envelope conditions (in correct timing) into the FOC yields:

$$\frac{1}{c} = \beta \left\{ \frac{1}{\tilde{c}} (\alpha A \tilde{k}^{\alpha - 1}) + \beta \gamma \frac{1}{\tilde{c}} (\alpha A \tilde{k}^{\alpha - 1})) + \gamma \frac{1}{c} (-1) \right\}$$

Note that we have substituted for c and $\tilde{c}$ back into the equation to simplify it and that we have obtained -1 from the derivative $\frac{\partial c}{\partial \tilde{k}}$. Now rearranging to obtain c on left hand side and $\tilde{c}$ on right hand side we get:

$$\frac{1}{c}(1 + \beta\gamma) = (1 + \beta\gamma)\beta \frac{1}{\tilde{c}} (\alpha A \tilde{k}^{\alpha - 1})$$

$$\frac{1}{c} = \beta \cdot \frac{\alpha A \tilde{k}^{\alpha-1}}{\tilde{c}}$$

Therefore after simplification we obtained the same result as in the problem without habit persistence. This is because the habit persistence takes the so-called 'additive separable' form!

In general, in steady state habit persistence does not have any effect, because in steady state the effect disappears (consumption is stable over time), hence I will not derive the expression here (see previous problem). Only effect is on the adjustment dynamics, but that's outside if the content of this part of the course.

□

Problem 3 (Bellman Equation: Multiple Constraints) Consider the following model of economic growth. The world is deterministic. Time is discrete. Representative dynasty maximises lifetime welfare given by

$$\sum_{t=0}^{\infty} \beta^t \left(\ln c_t - \eta l_t\right),$$

where h_0 is given, c_t is consumption, l_t is labor effort, $\eta > 0$, and discount factor $\beta \in (0,1)$. The household allocates its resources (disposable income) $\hat{y}_t$ between consumption and investment into human capital, e_t. Disposable income is given as

$$\hat{y}_t = (1-\tau) y_t, \tag{1}$$

where $\tau \in [0,1]$ is a proportional tax rate collected by the government. Pre-tax income of household is given by

$$y_t = h_t^{\lambda} l_t^{1-\lambda}, \tag{2}$$

where $0 < l < 1$. Human capital of the household accumulates according to

$$h_{t+1} = [(1+a)e_t]^{\gamma}, \tag{3}$$

where constant a is the proportional subsidy paid by the government to support human capital investment, and $0 < \gamma < 1$. As is seen from the description of the model, the only form of "savings" is possible in form of spending resources on producing human capital h_t. This could be justified if we assume that the output y_t consist of a perishable good. Current human capital h_t, does not influence future h_{t+1}, therefore, "human capital" can be thought of as a person's skills that fully depreciate within a period. Household takes parameters of the tax/subsidy τ and a as given.

1. Derive First Order Condition(s) and Envelope Theorem condition(s).

2. Derive the share of education expenditures in the disposable income, or "savings rate", which would prevail in the steady state:

$$s^* = \frac{e^*}{\hat{y}^*}$$

3. Derive steady state values of labor effort, consumption, and human capital.

Solution: We start by eliminating as much of the variables as possible. First note, that since household allocates it's resources (disposable income) $\hat{y}_t$ between consumption and investment into human capital, e_t, the following equation holds:

$$\hat{y}_t = c_t + e_t. \tag{4}$$

Therefore, plugging (4) into the LHS of equation (1), and also substituting for y_t from equation (2) into the RHS of (1) we get:

$$c_t + e_t = (1-\tau)h_t^{\lambda} l_t^{1-\lambda}.$$

We can also move equation (3) one period back and substitute to the equation above to get

$$c_t + e_t = (1-\tau)(1+a)^{\gamma\lambda} e_{t-1}^{\gamma\lambda} l_t^{1-\lambda}$$

or rewriting

$$c_t = (1-\tau)(1+a)^{\gamma\lambda} e_{t-1}^{\gamma\lambda} l_t^{1-\lambda} - e_t. \tag{5}$$

Now we can identify the remaining state and control variables:

- **states:** e_{t-1},
- **controls:** c_t, l_t, e_t.

We are now ready to set up the Bellman equation for the problem:

$$V(e_{t-1}) = \max_{e_t, l_t} \left\{ (\log c_t - \eta l_t) + \beta V(e_t) \right\}$$

(to avoid messy expressions we will keep c_t, even though we will not treat it as control variable anymore). First we take the F.O.C.s (since we have two control variables we will also have two F.O.C.s):

$$\frac{1}{c_t}(-1) + \beta V'(e_t) = 0, \tag{6}$$

$$\frac{1}{c_t}\Big((1-\tau)(1+a)^{\gamma\lambda} e_{t-1}^{\gamma\lambda} l_t^{-\lambda}(1-\lambda)\Big) - \eta = 0, \tag{7}$$

and the E.T. condition:

$$V'(e_{t-1}) = \frac{1}{c_t}\Big((1-\tau)\gamma\lambda(1+a)^{\gamma\lambda} e_{t-1}^{\gamma\lambda-1} l_t^{1-\lambda}\Big). \tag{8}$$

From now on assume that we are in the steady state. Denote by

$$l = l_t = l_{t-1},$$

$$c = c_t = c_{t-1},$$

$$e = e_t = e_{t-1}$$

the steady-state values of l, c and e respectively.

We plug in equation (8) into (6) to get:

$$\beta\frac{1}{c}(1-\tau)(1+a)^{\gamma\lambda}\gamma\lambda e^{\gamma\lambda-1}l^{1-\lambda} = \frac{1}{c}.$$

Using (5) (in the steady state) this simplifies to

$$\beta\gamma\lambda\frac{1}{e}(c+e) = 1,$$

or

$$s = \beta\gamma\lambda.$$

Rewriting equation (7)

$$\frac{c+e}{cl}(1-\lambda) = \eta,$$

and using the derived steady-state value for s we get

$$\frac{e}{cl}(1-\lambda) = \eta\beta\gamma\lambda \Longrightarrow l = \frac{l_t(\frac{1}{\beta\gamma\lambda} - 1r_t)(1-\lambda)}{\eta\beta\lambda\gamma}.$$

We now can use equation (5) and equation for s to solve for the steady-state values of e and c. $\square$