LNU
Spring 2017

Introduction to Dynamic Economic Models
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Exercise Session 1

Problem 1 (Bellman Equation: Basic) Consider social planner's problem of maximizing lifetime utility of the representative consumer:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

subject to the following constraints:

$$\begin{aligned} y_t &= c_t + i_t, \\ y_t &= f(k_t), \\ k_{t+1} &= (1-\delta)k_t + i_t, \\ k_{t+1} &\geq 0,\ c_t \geq 0, \\ k_0 &\text{ is given,} \end{aligned}$$

where c is consumption, i is investment, k is capital and output y is produced from capital using production function $f(\cdot)$; δ is the depreciation rate of capital.

1. Clearly identify state and control variables. Set up the Bellman equation for the problem (that is write the problem in the recursive form).
2. Derive First Order Conditions and Envelope Theorem conditions.
3. Using the equations from above derive Euler equation.
4. Derive expression for steady state level of capital and consumption in terms of parameters of the model. You can assume that $f(k) = k^\alpha$ for $\alpha < 1$.

Problem 2 (Bellman Equation: Multiple States) Consider the problem of consumer who seeks to solve

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t (\log c_t + \gamma \log c_{t-1}) \qquad \text{with} \quad 0 < \gamma < 1$$

subject to the following constraints:

$$k_{t+1} + c_t \leq A k_t^\alpha \quad A > 0, \quad \alpha \in (0,1)$$

$$c_t \leq 0, \quad k_t > 0, \quad k_0, c_{-1} \text{ given}$$

1. Clearly identify state and control variable(s).

2. Set up the Bellman equation for the problem (that is write the problem in the recursive form). *Hint:* Value function for the consumer will be the function of two variables.

3. Derive FOC, ET and EE.

4. Derive expression for steady state level of capital and consuption in terms of parameters of the model.

Problem 3 (Bellman Equation: Multiple Constraints) Consider the following model of economic growth. The world is deterministic. Time is discrete. Representative dynasty maximises lifetime welfare given by

$$\sum_{t=0}^{\infty} \beta^t \left(\ln c_t - \eta l_t\right),$$

where h_0 is given, c_t is consumption, l_t is labor effort, $\eta > 0$, and discount factor $\beta \in (0, 1)$. The household allocates its resources (disposable income) $\hat{y}_t$ between consumption and investment into human capital, e_t. Disposable income is given as

$$\hat{y}_t = (1 - \tau) y_t, \tag{1}$$

where $\tau \in [0, 1]$ is a proportional tax rate collected by the government. Pre-tax income of household is given by

$$y_t = h_t^{\lambda} l_t^{1-\lambda}, \tag{2}$$

where $0 < l < 1$. Human capital of the household accumulates according to

$$h_{t+1} = [(1 + a)e_t]^{\gamma}, \tag{3}$$

where constant a is the proportional subsidy paid by the government to support human capital investment, and $0 < \gamma < 1$. As is seen from the description of the model, the only form of "savings" is possible in form of spending resources on producing human capital h_t. This could be justified if we assume that the output y_t consist of a perishable good. Current human capital h_t, does not influence future h_{t+1}, therefore, "human capital" can be thought of as a person's skills that fully depreciate within a period. Household takes parameters of the tax/subsidy τ and a as given.

1. Derive First Order Condition(s) and Envelope Theorem condition(s).

2. Derive the share of education expenditures in the disposable income, or "savings rate", which would prevail in the steady state:

$$s^* = \frac{e^*}{\hat{y}^*}$$

3. Derive steady state values of labor effort, consumption, and human capital.