Formal Languages, Automata and Codes

Oleg Gutik

Lecture 11

4.1 Closure Properties of Regular Languages

Consider the following question: Given two regular languages L_1 and L_2, is their union also regular? In specific instances, the answer may be obvious, but here we want to address the problem in general. Is it true for all regular L_1 and L_2? It turns out that the answer is yes, a fact we express by saying that the family of regular languages is **closed** under union. We can ask similar questions about other types of operations on languages; this leads us to the study of the closure properties of languages in general.

Closure properties of various language families under different operations are of considerable theoretical interest. At first sight, it may not be clear what practical significance these properties have. Admittedly, some of them have very little, but many results are useful. By giving us insight into the general nature of language families, closure properties help us answer other, more practical questions. We shall see instances of this later in this couse of lectures.

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We begin by looking at the closure of regular languages under the common set operations, such as union and intersection.

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Proof. If L_1 and L_2 are regular, then there exist regular expressions r_1 and r_2 such that $L_1 = L(r_1)$ and $L_2 = L(r_2)$. By definition, $r_1 + r_2$, $r_1 r_2$, and r_1^* are regular expressions denoting the languages $L_1 \cup L_2$, $L_1 L_2$, and L_1^*, respectively. Thus, closure under union, concatenation, and star-closure is immediate.

To show closure under complementation, let $M = (Q, \Sigma, \delta, q_0, F)$ be a dfa that accepts L_1. Then the dfa

$$\widehat{M} = (Q, \Sigma, \delta, q_0, Q - F)$$

accepts $\overline{L_1}$. This is rather straightforward. Note that in the definition of a dfa, we assumed δ^* to be a total function, so that $\delta^*(q_0, w)$ is defined for all $w \in \Sigma^*$. Consequently either $\delta^*(q_0, w)$ is a final state, in which case $w \in L$, or $\delta^*(q_0, w) \in Q - F$ and $w \in L$.

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If L_1 and L_2 are regular languages, then so are $L_1 \cup L_2$, $L_1 \cap L_2$, $L_1 L_2$, $\overline{L_1}$, and L_1^*. We say that the family of regular languages is closed under union, intersection, concatenation, complementation, and star-closure.

Proof. If L_1 and L_2 are regular, then there exist regular expressions r_1 and r_2 such that $L_1 = L(r_1)$ and $L_2 = L(r_2)$. By definition, $r_1 + r_2$, $r_1 r_2$, and r_1^* are regular expressions denoting the languages $L_1 \cup L_2$, $L_1 L_2$, and L_1^*, respectively. Thus, closure under union, concatenation, and star-closure is immediate.

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Demonstrating closure under intersection takes a little more work. Let $L_1 = L(M_1)$ and $L_2 = L(M_2)$, where $M_1 = (Q, \Sigma, \delta_1, q_0, F_1)$ and $M_2 = (P, \Sigma, \delta_2, p_0, F_2)$ are dfa's. We construct from M_1 and M_2 a combined automaton $\widehat{M} = (\widehat{Q}, \Sigma, \widehat{\delta}, (q_0, p_0), \widehat{F})$, whose state set $\widehat{Q} = Q \times P$ consists of pairs (q_i, p_j), and whose transition function $\widehat{\delta}$ is such that $\widehat{M}$ is in state (q_i, p_j) whenever M_1 is in state q_i and M_2 is in state p_j. This is achieved by taking

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The following example is a variation on the same idea.

Example 4.1

Show that the family of regular languages is closed under difference. In other words, we want to show that if L_1 and L_2 are regular, then $L_1 - L_2$ is necessarily regular also.

The needed set identity is immediately obvious from the definition of a set difference, namely

$$L_1 - L_2 = L_1 \cap \overline{L_2}.$$

The fact that L_2 is regular implies that $\overline{L_2}$ is also regular. Then, because of the closure of regular languages under intersection, we know that $L_1 \cap \overline{L_2}$ is regular, and the argument is complete.

A variety of other closure properties can be derived directly by elementary arguments.

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A variety of other closure properties can be derived directly by elementary arguments.

The following example is a variation on the same idea.

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Show that the family of regular languages is closed under difference. In other words, we want to show that if L_1 and L_2 are regular, then $L_1 - L_2$ is necessarily regular also.

The needed set identity is immediately obvious from the definition of a set difference, namely

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Theorem 4.2

The family of regular languages is closed under reversal.

Proof. Suppose that L is a regular language. We then construct an nfa with a single final state for it. In the previous lectures we show that this is always possible. In the transition graph for this nfa we make the initial vertex a final vertex, the final vertex the initial vertex, and reverse the direction on all the edges. It is a fairly straightforward matter to show that the modified nfa accepts w^R if and only if the original nfa accepts w. Therefore, the modified nfa accepts L^R, proving closure under reversal. ■

Closure under Other Operations

In addition to the standard operations on languages, one can define other operations and investigate closure properties for them. There are many such results; we select only two typical ones.

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is called a *homomorphism*. In words, a homomorphism is a substitution in which a single letter is replaced with a string. The domain of the function h is extended to strings in an obvious fashion; if
$$w = a_1 a_2 \cdots a_n,$$
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If L is a language on Σ, then its homomorphic image is defined as
$$h(L) = \{h(w) \colon w \in L\}.$$

Example 4.2

Let $\Sigma = \{a, b\}$ and $\Gamma = \{a, b, c\}$ and define h by
$$h(a) = ab,$$
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$$h(L) = \{h(w)\colon w \in L\}.$$

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Let $\Sigma = \{a, b\}$ and $\Gamma = \{a, b, c\}$ and define h by
$$h(a) = ab,$$
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Suppose Σ and Γ are alphabets. Then a function

$$h\colon \Sigma \to \Gamma^*$$

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Take $\Sigma = \{a, b\}$ and $\Gamma = \{b, c, d\}$. Define h by

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If L is the regular language denoted by

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then

$$r_1 = (dbcc + (bdc)^*)\,(dbccdbcc)^*$$

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The general result on the closure of regular languages under any homomorphism follows from this example in an obvious manner.

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Let h be a homomorphism. If L is a regular language, then its homomorphic image $h(L)$ is also regular. The family of regular languages is therefore closed under arbitrary homomorphisms.

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Theorem 4.3

Let h be a homomorphism. If L is a regular language, then its homomorphic image $h(L)$ is also regular. The family of regular languages is therefore closed under arbitrary homomorphisms.

Proof. Let L be a regular language denoted by some regular expression r. We find $h(r)$ by substituting $h(a)$ for each symbol $a \in \Sigma$ of r. It can be shown directly by an appeal to the definition of a regular expression that the result is a regular expression. It is equally easy to see that the resulting expression denotes $h(L)$. All we need to do is to show that for every $w \in L(r)$, the corresponding $h(w)$ is in $L(h(r))$ and conversely that for every v in $L(h(r))$ there is a word w in L, such that $v = h(w)$. Leaving the details as an exercise, we claim that $h(L)$ is regular. ∎

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Example 4.4

If

$$L_1 = \{a^n b^m : n \geqslant 1, m \geqslant 0\} \cup \{ba\}$$

and

$$L_2 = \{b^m : m \geqslant 1\},$$

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The strings in L_2 consist of one or more b's. Therefore, we arrive at the answer by removing one or more b's from those strings in L_1 that terminate with at least one b.

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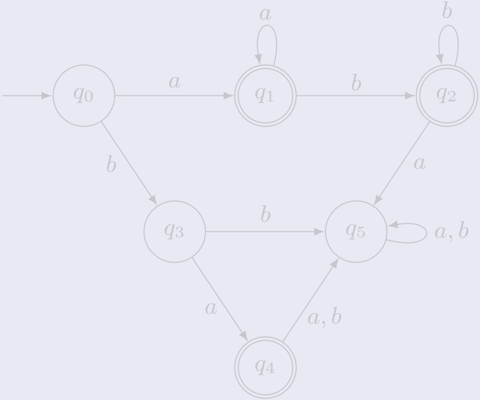
Example 4.4 (continuation)

Since an automaton for L_1/L_2 must accept any prefix of strings in L_1, we shall try to modify M_1 so that it accepts x if there is any y satisfying (1).

$$L_1/L_2 = \{x\colon xy \in L_1 \text{ for some } y \in L_2\}. \tag{1}$$

The difficulty comes in finding whether there is some y such that $xy \in L_1$ and $y \in L_2$. To solve it, we determine, for each $q \in Q$, whether there is a walk to a final state labeled v such that $v \in L_2$. If this is so, any x such that $\delta(q_0, x) = q$ will be in L_1/L_2. We modify the automaton accordingly to make q a final state.

Example 4.4 (continuation)



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The difficulty comes in finding whether there is some y such that $xy \in L_1$ and $y \in L_2$. To solve it, we determine, for each $q \in Q$, whether there is a walk to a final state labeled v such that $v \in L_2$. If this is so, any x such that $\delta(q_0, x) = q$ will be in L_1/L_2. We modify the automaton accordingly to make q a final state.

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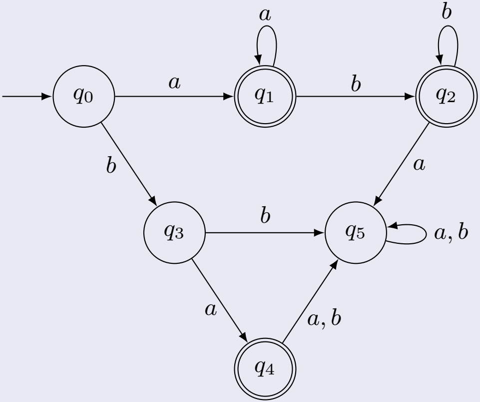

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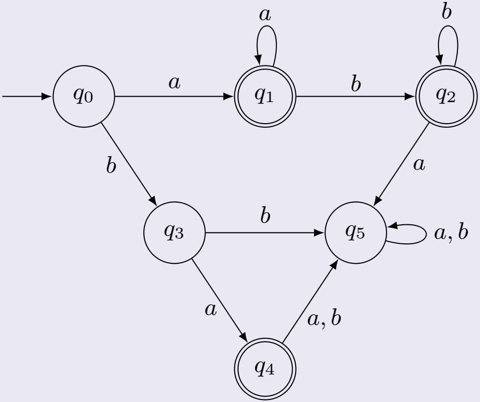

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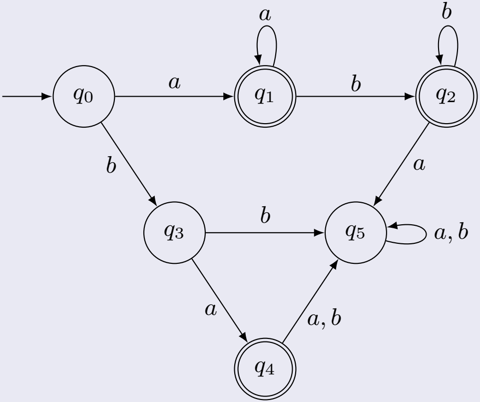

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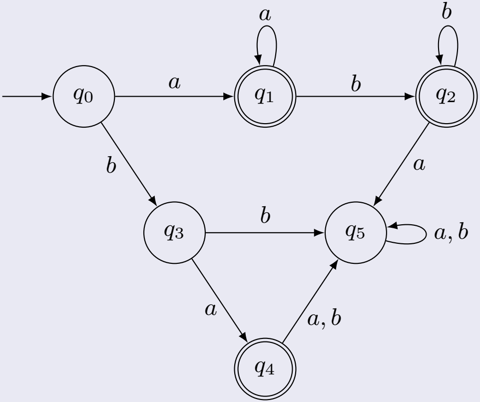

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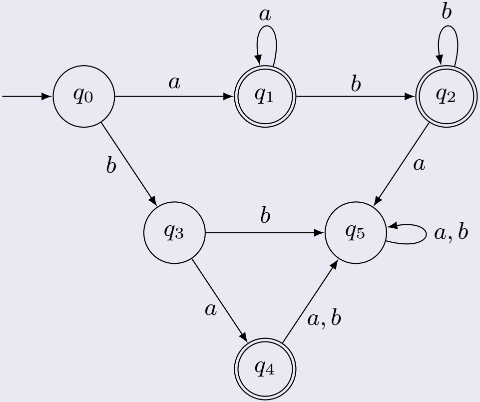

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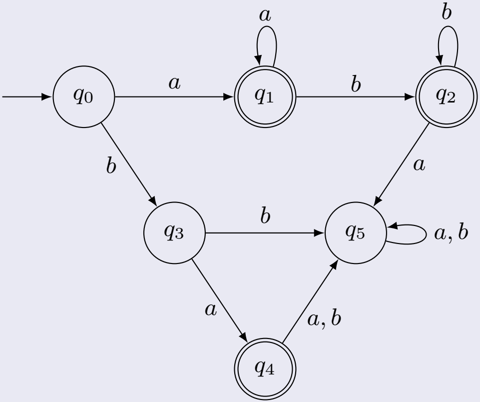

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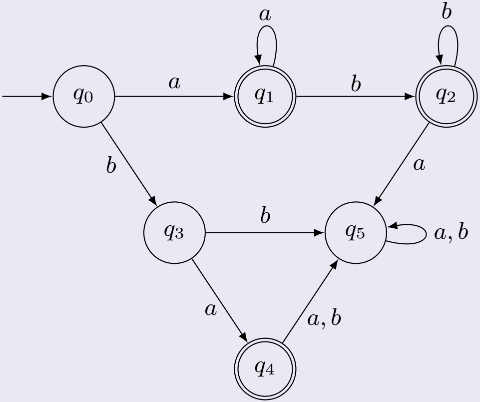

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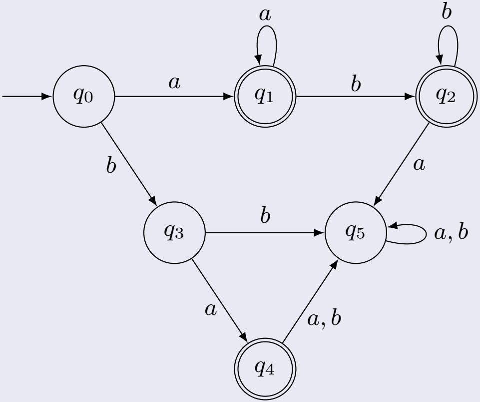

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Check it to see that the construction works. The idea is generalized in the next theorem.

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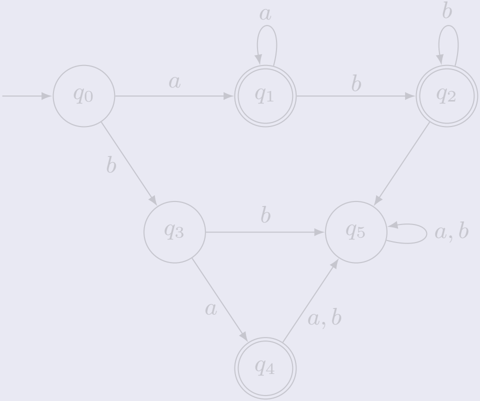

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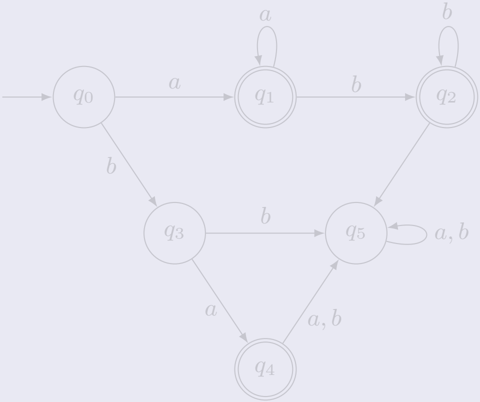

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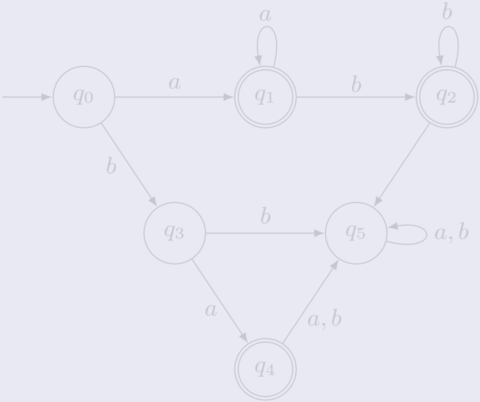

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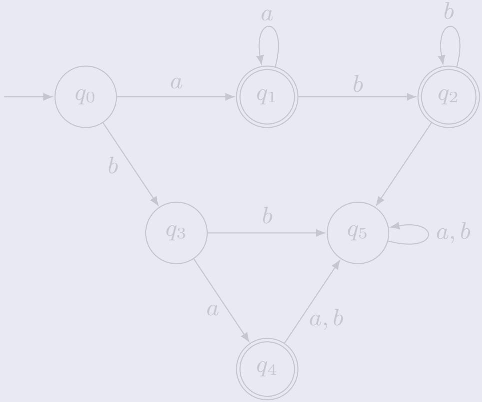

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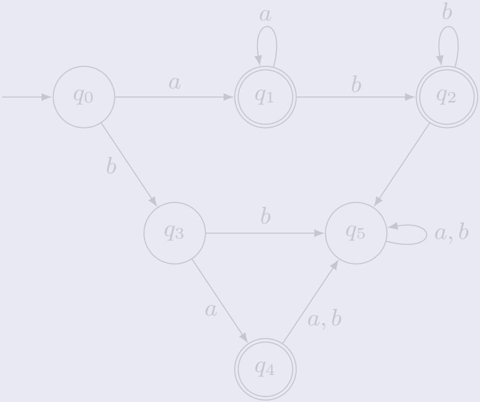

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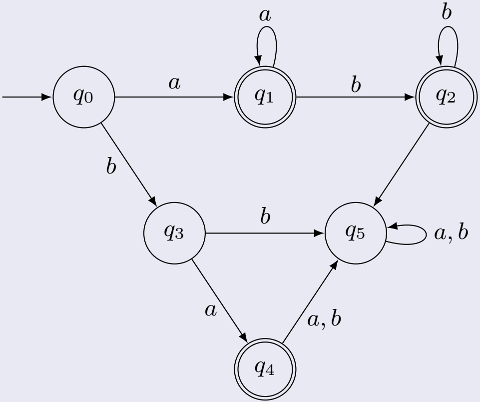

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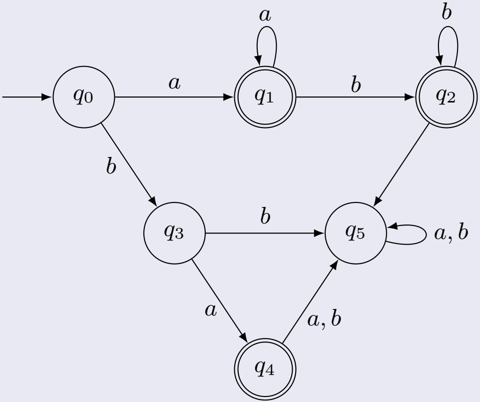

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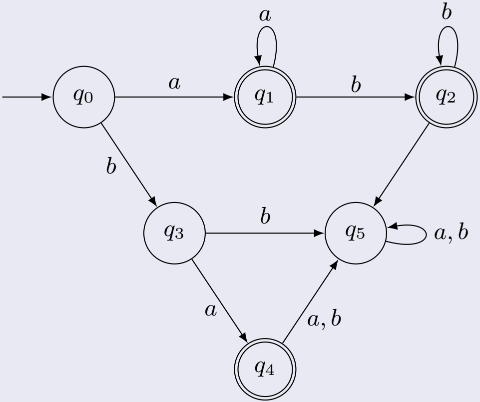

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Theorem 4.4

If L_1 and L_2 are regular languages, then L_1/L_2 is also regular. We say that the family of regular languages is closed under right quotient with a regular language.

Proof. Let $L_1 = L(M)$, where $M = (Q, \Sigma, \delta, q_0, F)$ is a dfa. We construct another dfa $\widehat{M} = (Q, \Sigma, \delta, q_0, \widehat{F})$ as follows. For each $q_i \in Q$, determine if there exists a word $y \in L_2$ such that

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Therefore, by construction, $q \in \widehat{F}$, and $\widehat{M}$ accepts x because $\delta^*(q_0, x)$ is in $\widehat{F}$.

Conversely, for any x accepted by $\widehat{M}$, we have

$$\delta^*(q_0, x) = q \in \widehat{F}.$$

But again by construction, this implies that there exists a word $y \in L_2$ such that $\delta^*(q, y) \in F$. Therefore, xy is in L_1, and x is in L_1/L_2. We therefore conclude that

$$L(\widehat{M}) = L_1/L_2,$$

and from this that L_1/L_2 is regular. ■

To prove that $L(\widehat{M}) = L_1/L_2$, let x be any element of L_1/L_2. Then there must be a word $y \in L_2$ such that $xy \in L_1$. This implies that

$$\delta^*(q_0, xy) \in F,$$

so that there must be some $q \in Q$ such that

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Example 4.5

Find L_1/L_2 for

$$L_1 = L(a^*baa^*),$$
$$L_2 = L(ab^*).$$

We first find a dfa that accepts L_1. This is easy, and a solution is given in the following Figure.

The example is simple enough so that we can skip the formalities of the construction.

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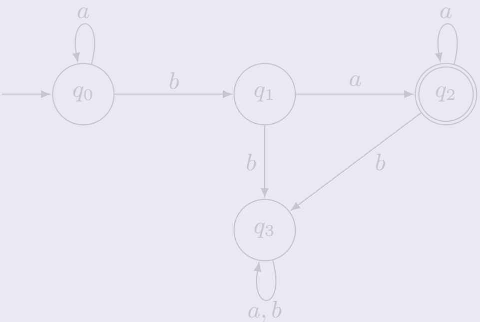

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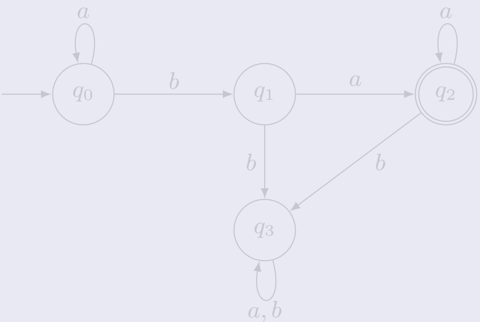

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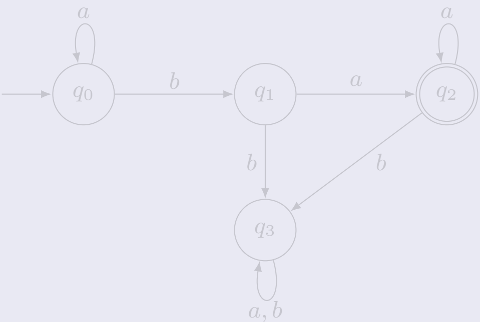

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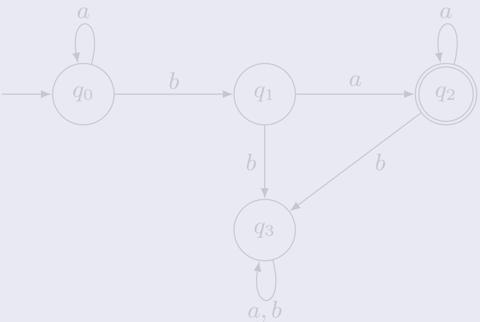

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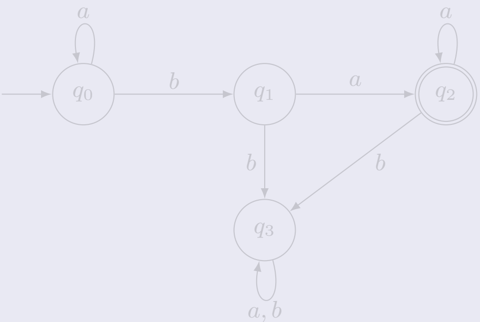

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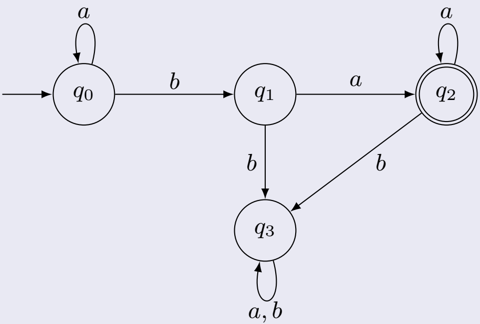

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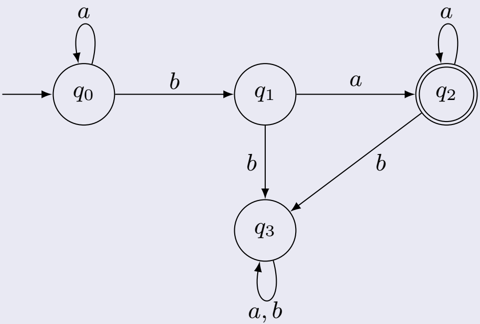

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Example 4.5 (continuation)

From the graph in the previous Figure it is quite evident that

$$L(M_0) \cap L_2 = \varnothing,$$

$$L(M_1) \cap L_2 = \{a\} \neq \varnothing,$$

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Therefore, the automaton accepting L_1/L_2 is determined. The result is shown in the Figure.

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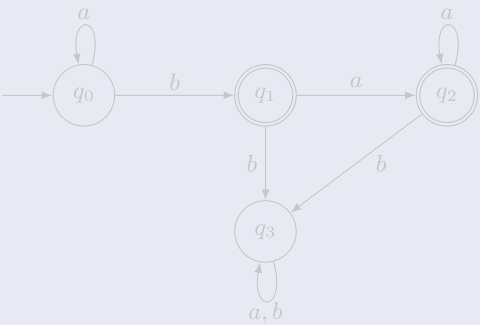

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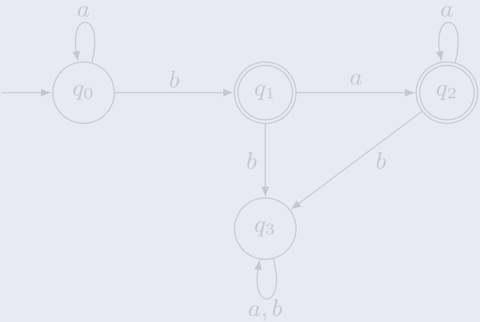

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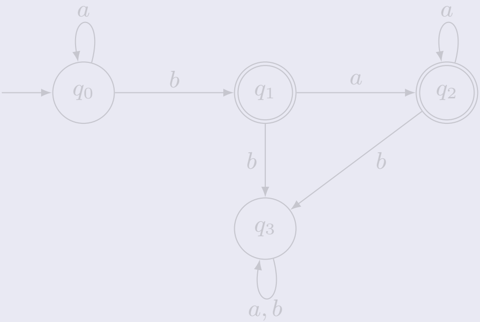

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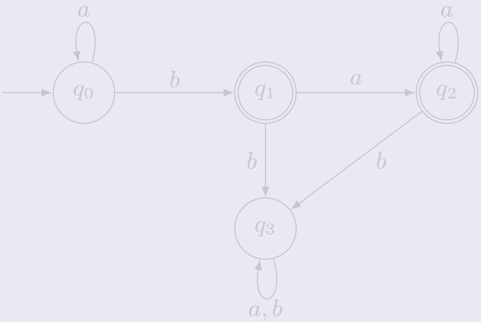

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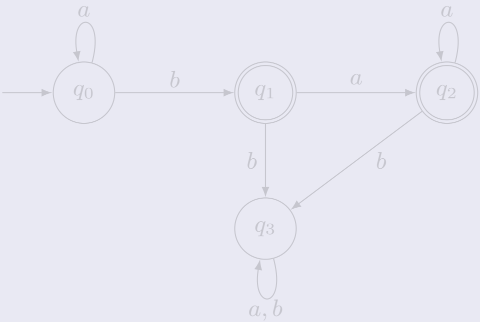

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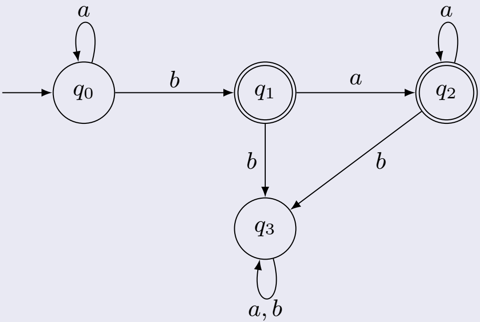

Example 4.5 (continuation)

It accepts the language denoted by the regular expression of $a^*b + a^*baa^*$, which can be simplified to a^*ba^*. Thus $L_1/L_2 = L(a^*ba^*)$.

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Thank You for attention!