Границя функції: друга чудова границя

Практичне заняття 11

Друга чудова границя

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$$

Або узагальнення: $\lim\limits_{\alpha(x)\to 0}\left(1+\alpha(x)\right)^{\frac{1}{\alpha(x)}}=e$.

2.168 (Демид. 506(а)) $\lim\limits_{x\to 0}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}=\left(\frac{1}{2}\right)^{1}=\boxed{\frac{1}{2}}$.

2.169 (Демид. 506(б)) $\lim\limits_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}=\lim\limits_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{(1-\sqrt{x})(1+\sqrt{x})}}=$

$=\lim\limits_{x\to 1}\left(\frac{1+x}{2+x}\right)^{\frac{1}{1+\sqrt{x}}}=\left(\frac{2}{3}\right)^{\frac{1}{2}}=\boxed{\sqrt{\frac{2}{3}}}$.

2.170 (Демид. 506(в)) $\lim\limits_{x\to+\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1-\sqrt{x}}{1-x}}=\lim\limits_{x\to+\infty}\left(\frac{1+x}{2+x}\right)^{\frac{1}{1+\sqrt{x}}}=$

$\lim\limits_{x\to+\infty}\frac{1+x}{2+x}=\lim\limits_{x\to+\infty}\frac{\frac{1}{x}+1}{\frac{2}{x}+1}=1$

$=1^{0}=\boxed{1}$.

2.173

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x=\lim_{x\to\infty}\left(\frac{(x+1)-1}{x+1}\right)^x=\lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^x=$$

$$=\{1^\infty\}=\lim_{x\to\infty}\left(1-\frac{1}{x+1}\right)^{-(x+1)\cdot\frac{x}{-(x+1)}}=$$

$$=\lim_{x\to\infty}\left(\left(1-\frac{1}{x+1}\right)^{-(x+1)}\right)^{-\frac{x}{x+1}}=e^{-1}=\boxed{\frac{1}{e}}.$$

2.178 (Demidowitsch 512)

$$\lim_{x\to\infty}\left(\frac{x^2+1}{x^2-2}\right)^{x^2}=\{1^\infty\}=\lim_{x\to\infty}\left(\frac{(x^2-2)+3}{x^2-2}\right)^{x^2}=$$

$$=\lim_{x\to\infty}\left(1+\frac{3}{x^2-2}\right)^{\frac{x^2-2}{3}\cdot\frac{3x^2}{x^2-2}}=e^{\lim\limits_{x\to\infty}\frac{3x^2}{x^2-2}}=e^{\lim\limits_{x\to\infty}\frac{3}{1-\frac{2}{x^2}}}=$$

$$=\boxed{e^3}.$$

2.180 (Demid. 514)

$$\lim_{x\to 0} \sqrt[x]{1-2x} = \lim_{x\to 0} (1-2x)^{\frac{1}{x}} = \{1^{\infty}\} = \lim_{x\to 0} (1-2x)^{-\frac{1}{2x}\cdot\frac{-2x}{x}} =$$

$$= \lim_{x\to 0} \left((1-2x)^{-\frac{1}{2x}}\right)^{-2} = e^{-2} = \boxed{\frac{1}{e^2}}.$$

2.183 (Demid. 517)

$$\lim_{x\to 0} (1+x^2)^{ctg^2 x} = \lim_{x\to 0} (1+x^2)^{\frac{\cos^2 x}{\sin^2 x}} = \{1^{\infty}\} =$$

$$= \lim_{x\to 0} (1+x^2)^{\frac{1}{x^2}\cdot\frac{x^2}{\sin^2 x}\cdot\cos^2 x} = e^1 = \boxed{e}.$$

2.190 (Demid. 523)

$$\lim_{x\to\frac{\pi}{2}} (\sin x)^{tgx} = \lim_{x\to\frac{\pi}{2}} (\sin x)^{\frac{\sin x}{\cos x}} = \{1^\infty\} =$$

$$= \lim_{x\to\frac{\pi}{2}} \left(1+(\sin x-1)\right)^{\frac{\sin x}{\cos x}} =$$

$$= \lim_{x\to\frac{\pi}{2}} \left(1+(\sin x-1)\right)^{\frac{1}{\sin x-1}\cdot\frac{\sin x-1}{1}\cdot\frac{\sin x}{\cos x}} =$$

$$= \lim_{x\to\frac{\pi}{2}} \left(\left(1+(\sin x-1)\right)^{\frac{1}{\sin x-1}}\right)^{\frac{\sin x-1}{\cos x}\cdot\sin x} = e^0 = \boxed{1}$$

- $\lim\limits_{x\to\frac{\pi}{2}} \dfrac{\sin x-1}{\cos x} = \left\{\begin{matrix}\frac{\pi}{2}-x=y\\ x=\frac{\pi}{2}-y\end{matrix}\right\} = \lim\limits_{y\to 0} \dfrac{\sin\left(\frac{\pi}{2}-y\right)-1}{\cos\left(\frac{\pi}{2}-y\right)} =$

$$= \lim_{y\to 0} \frac{\cos y-1}{\sin y} = -\lim_{y\to 0} \frac{y^2/2}{y} = -\frac{1}{2}\lim_{y\to 0} y = 0.$$

2.197 (Демид. 530)

$$\lim_{x\to+\infty} x\left(\ln(x+1)-\ln x\right) = \lim_{x\to+\infty} x \ln\frac{x+1}{x} =$$

$$= \lim_{x\to+\infty} x\cdot\ln\left(1+\frac{1}{x}\right) = \lim_{x\to+\infty} \ln\left(1+\frac{1}{x}\right)^{x} = \ln e = \boxed{1}.$$

$\left(1+\frac{1}{x}\right)^{x} \to e$

2.200 (Демид. 533)

$$\lim_{x\to+\infty} \frac{\ln(x^2-x+1)}{\ln(x^{10}+x+1)} = \lim_{x\to+\infty} \frac{\ln\left(x^2\cdot\left(1-\frac{1}{x}+\frac{1}{x^2}\right)\right)}{\ln\left(x^{10}\cdot\left(1+\frac{1}{x^9}+\frac{1}{x^{10}}\right)\right)} =$$

$$= \lim_{x\to+\infty} \frac{\ln x^2+\ln\left(1-\frac{1}{x}+\frac{1}{x^2}\right)}{\ln x^{10}+\ln\left(1+\frac{1}{x^5}+\frac{1}{x^{10}}\right)} = \lim_{x\to+\infty} \frac{2\ln x+0}{10\ln x+0} = \boxed{\frac{1}{5}}.$$

2.203

$$\lim_{x\to+\infty}\frac{\ln(1+2^x)}{\ln(1+5^{2x})}=\lim_{x\to+\infty}\frac{\ln\left(2^x\cdot\left(\frac{1}{2^x}+1\right)\right)}{\ln\left(5^{2x}\cdot\left(\frac{1}{5^{2x}}+1\right)\right)}=$$

$$=\lim_{x\to+\infty}\frac{x\ln 2+\ln\left(2^{-x}+1\right)}{2x\ln 5+\ln\left(5^{-2x}+1\right)}=$$

$$=\lim_{x\to+\infty}\frac{\ln 2+\frac{\ln\left(2^{-x}+1\right)}{x}}{2\ln 5+\frac{\ln\left(5^{-2x}+1\right)}{x}}=\boxed{\frac{\ln 2}{5\ln 5}}$$

2.207 (Демид. 540.1)

$$\lim_{x\to 0}\frac{\ln\left(nx+\sqrt{1-n^2x^2}\right)}{\ln\left(x+\sqrt{1-x^2}\right)}=\left\{\frac{\ln 1}{\ln 1}=\frac{0}{0}\right\}=\left[\text{зведемо до } \lim_{\alpha(x)\to 0}\ln\left(1+\alpha(x)\right)^{\frac{1}{\alpha(x)}}=1\right]$$

$$=\lim_{x\to 0}\frac{\ln\left(1+\left(nx-1+\sqrt{1-n^2x^2}\right)\right)}{\ln\left(1+\left(x-1+\sqrt{1-x^2}\right)\right)}=$$

$$=\lim_{x\to 0}\frac{\left(nx-1+\sqrt{1-n^2x^2}\right)\overbrace{\ln\left(1+\left(nx-1+\sqrt{1-n^2x^2}\right)\right)^{\frac{1}{nx-1+\sqrt{1-n^2x^2}}}}^{\to 1}}{\left(x-1+\sqrt{1-x^2}\right)\underbrace{\ln\left(1+\left(x-1+\sqrt{1-x^2}\right)\right)^{\frac{1}{x-1+\sqrt{1-x^2}}}}_{\to 1}}=$$

$$=\lim_{x\to 0}\frac{nx-1+\sqrt{1-n^2x^2}}{x-1+\sqrt{1-x^2}}=\lim_{x\to 0}\frac{(nx-1)^2-(1-n^2x^2)}{\left(nx-1-\sqrt{1-n^2x^2}\right)}\cdot\frac{x-1-\sqrt{1-x^2}}{(x-1)^2-(1-x^2)}$$

$$=\lim_{x\to 0}\frac{2n^2x^2-2nx}{2x^2-2x}\cdot\frac{x-1-\sqrt{1-x^2}}{nx-1-\sqrt{1-n^2x^2}}=\frac{-2}{-2}\lim_{x\to 0}\frac{n^2x-n}{x-1}=\boxed{n}$$

$$\lim_{x\to 0}\frac{a^x-1}{x}=\ln a \quad\Rightarrow\quad a^x-1\sim x\cdot\ln a,\ x\to 0.$$

$$\lim_{x\to 0}\frac{(1+x)^\alpha-1}{x}=\alpha \quad\Rightarrow\quad (1+x)^\alpha-1\sim \alpha x,\ x\to 0.$$

№2.209 (Демид. 545) $\lim\limits_{x\to 0}\left(\frac{1+x\cdot 2^x}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}}=\{1^\infty\}=$

$$=\lim_{x\to 0}\left(\frac{(1+x\cdot 3^x)+(x\cdot 2^x-x\cdot 3^x)}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}}=\lim_{x\to 0}\left(1+\frac{x\cdot(2^x-3^x)}{1+x\cdot 3^x}\right)^{\frac{1}{x^2}}=$$

$$=\lim_{x\to 0}\left(1+\frac{x(2^x-3^x)}{1+x\cdot 3^x}\right)^{\frac{1+x\cdot 3^x}{x(2^x-3^x)}\cdot\frac{2^x-3^x}{(1+x\cdot 3^x)\cdot x}}=$$

$$=e^{\lim\limits_{x\to 0}\frac{2^x-3^x}{(1+x\cdot 3^x)\cdot x}}=e^{\lim\limits_{x\to 0}\frac{2^x-3^x}{x}}=e^{\lim\limits_{x\to 0}\left(\frac{2^x-1}{x}+\frac{1-3^x}{x}\right)}=$$

$$=e^{\ln 2-\ln 3}=e^{\ln\frac{2}{3}}=\boxed{\frac{2}{3}}.$$

2 211

$$\frac{a^x - 1}{x} \to \ln a,\ x \to 0$$

$$\lim_{x\to 0} \frac{e^{\sin 2x} - e^{\sin x}}{x} = \lim_{x\to 0}\left(\frac{e^{\sin 2x} - 1}{x} + \frac{1 - e^{\sin x}}{x}\right) =$$

$$= \lim_{x\to 0}\left(\frac{e^{\sin 2x} - 1}{\sin 2x} \cdot \frac{\sin 2x}{x} - \frac{e^{\sin x} - 1}{\sin x} \cdot \frac{\sin x}{x}\right) =$$

$\frac{e^{\sin 2x} - 1}{\sin 2x} \to 1$, $\frac{e^{\sin x} - 1}{\sin x} \to 1$

$$= 1 \cdot \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot 2 - 1 \cdot 1 = 2 - 1 = \boxed{1}$$

$\frac{\sin 2x}{2x} \to 1$

Домашнє завдання :

із задач 2.171- 2.213 зробити будь-яких 10 прикладів з тих, які ми не робили