# Границя функції: перша чудова границя

Практичне заняття 10

# Перша чудова границя.

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

**2.99** (Демид. №471)

$$\lim_{x\to 0}\frac{\sin 7x}{x}=\lim_{x\to 0}\frac{\sin 7x}{7x}\cdot 7=\boxed{7},$$ бо

$$\lim_{\varphi(x)\to 0}\frac{\sin\varphi(x)}{\varphi(x)}=1$$

**2.100** (Демид. 472)

$$\lim_{x\to\infty}\frac{\sin x}{x}=0,$$ бо $$\left|\frac{\sin x}{x}\right|\le\frac{1}{|x|}\to 0,\ x\to+\infty.$$

**2.102** (Демидович 474)

$$\lim_{x\to 0}\frac{1-\cos x}{x^2}=\lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}=\lim_{x\to 0}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\cdot\frac{1}{2}=\boxed{\frac{1}{2}}.$$

$$\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=2\cos^2\alpha-1=1-2\sin^2\alpha$$

**2.103** (Демидович 474.1)

$$\lim_{x\to 0}\frac{\operatorname{tg}x}{x}=\lim_{x\to 0}\frac{\sin x}{\cos x}\cdot\frac{1}{x}=\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x}=\boxed{1}.$$

**2.106** (Демид. 476) $\lim\limits_{x\to 0}\dfrac{\sin 5x-\sin 3x}{\sin x}=\lim\limits_{x\to 0}\dfrac{2\sin x\cdot\cos 4x}{\sin x}=$

$$\sin\alpha-\sin\beta=2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha+\beta}{2}$$

$=\boxed{2}$.

**2.108** (Demid. 479)

$$\lim_{x\to\frac{\pi}{4}} tg\,2x\cdot tg\left(\frac{\pi}{4}-x\right)=\left\{\begin{array}{c}\text{zamena}\\ y=\frac{\pi}{4}-x\\ x=\frac{\pi}{4}-y\end{array}\right\}=\lim_{y\to 0} tg\,2\cdot\left(\frac{\pi}{4}-y\right)\cdot tg\,y=$$

$$=\lim_{y\to 0} tg\left(\frac{\pi}{2}-2y\right)\cdot tg\,y=\lim_{y\to 0} ctg\,2y\cdot tg\,y=$$

$$=\lim_{y\to 0}\frac{\cos 2y}{\sin 2y}\cdot\frac{\sin y}{\cos y}=\lim_{y\to 0}\frac{\cos 2y}{\cos y}\cdot\frac{2y}{\sin 2y}\cdot\frac{1}{2}\cdot\frac{\sin y}{y}=$$

$\frac{2y}{\sin 2y}\to 1$, $\frac{\sin y}{y}\to 1$

$$=\boxed{\frac{1}{2}}.$$

**2.116**

$$\lim_{x\to 1}\frac{\sin(1-x)}{\sqrt{x}-1}=\lim_{x\to 1}\frac{\sin(1-x)}{(\sqrt{x}-1)(\sqrt{x}+1)}\cdot(\sqrt{x}+1)=$$

$$=\lim_{x\to 1}\frac{\sin(1-x)}{x-1}\cdot(\sqrt{x}+1)=-\lim_{x\to 1}\frac{\sin(1-x)}{1-x}\cdot(\sqrt{x}+1)=\boxed{-2}.$$

$\frac{\sin(1-x)}{1-x}\to 1$

**2.122**

$$\lim_{x\to 0}\frac{\sqrt{4+x}-2}{\sin 5x}=\lim_{x\to 0}\frac{(\sqrt{4+x}-2)(\sqrt{4+x}+2)}{\sin 5x\cdot(\sqrt{4+x}+2)}=$$

$$=\lim_{x\to 0}\frac{5x}{\sin 5x}\cdot\frac{1}{5}\cdot\frac{1}{\sqrt{4+x}+2}=\frac{1}{5}\cdot\frac{1}{4}=\boxed{\frac{1}{20}}.$$

**2.125**

$$\lim_{x\to 0}\frac{1-\cos 2x+x\sin x}{\sin^2 x}=\lim_{x\to 0}\frac{2\sin^2 x+x\cdot\sin x}{\sin^2 x}=$$

$$=\lim_{x\to 0}\frac{2\sin x+x}{\sin x}=\lim_{x\to 0}\left(2+\frac{x}{\sin x}\right)=2+1=\boxed{3}.$$

**2.128**

$$\lim_{x\to 0}\frac{\arcsin 3x}{3x}=\left\{\begin{array}{c}\text{zamena}\\ \arcsin 3x=y\\ 3x=\sin y\end{array}\right\}=\lim_{y\to 0}\frac{y}{\sin y}=\boxed{1}$$

Важливі еквівалентності:

- $\sin x \sim x, \quad x \to 0$
- $1 - \cos x \sim \frac{x^2}{2}, \quad x \to 0$
- $\operatorname{tg} x \sim x, \quad x \to 0$
- $\arcsin x \sim x, \quad x \to 0$
- $\operatorname{arctg} x \sim x, \quad x \to 0$.

**2.134** $$\lim_{x \to 0} \frac{x^3 \operatorname{ctg}^2 3x}{\sin^2 9x} = \lim_{x \to 0} \frac{x^3}{\sin 9x \cdot \operatorname{tg}^2 3x} =$$

$$= \lim_{x \to 0} \frac{x^3}{9x \cdot (3x)^2} = \frac{1}{9 \cdot 9} = \boxed{\frac{1}{81}}.$$

**2.137** $$\lim_{x \to 0} \frac{\operatorname{arctg} x}{3x} = \lim_{x \to 0} \frac{x}{3x} = \boxed{\frac{1}{3}}.$$

**2.142**

$$\lim_{x\to\frac{\pi}{2}}\left(2x\cdot tgx-\frac{\pi}{\cos x}\right)\overset{\{\infty-\infty\}}{=}\lim_{x\to\frac{\pi}{2}}\frac{2x\cdot\sin x-\pi}{\cos x}=$$

$$=\left\{\begin{array}{l}\text{замена}\\ x-\frac{\pi}{2}=y\\ x=y+\frac{\pi}{2}\end{array}\right\}=\lim_{y\to 0}\frac{2\left(y+\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}+y\right)-\pi}{\cos\left(\frac{\pi}{2}+y\right)}=$$

$$=\lim_{y\to 0}\frac{2\left(y+\frac{\pi}{2}\right)\cos y-\pi}{-\sin y}=\lim_{y\to 0}\frac{2y\cos y+\pi\cos y-\pi}{-\sin y}=$$

$$=\lim_{y\to 0}\left(\frac{y\cdot 2\cos y}{-\sin y}\to 1-\frac{\pi(1-\cos y)}{-\sin y}\right)=$$

$$=-2+\pi\lim_{y\to 0}\frac{y^2/2}{y}=-2+0=\boxed{-2}$$

2.147

$$\lim_{x\to 0}\frac{\cos(a+2x)-2\cos(a+x)+\cos a}{x^2}=\left\{\frac{0}{0}\right\}=$$

$$=\lim_{x\to 0}\frac{2\cos\frac{a+2x+a}{2}\cos\frac{a+2x-a}{2}-2\cos(a+x)}{x^2}=$$

$$=\lim_{x\to 0}\frac{2\cos(a+x)\cos x-2\cos(a+x)}{x^2}=\lim_{x\to 0}\left(\frac{-2\cos(a+x)}{1}\cdot\right.$$

$$\left.\cdot\frac{1-\cos x}{x^2}\right)=-2\cos a\cdot\frac{1}{2}=\boxed{-\cos a}$$

**2.151**

$$\lim_{x\to\frac{\pi}{6}} \frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1} = \left\{\frac{0}{0}\right\} =$$

$$= \left\{\begin{array}{l}\text{zamena}\\ t = \sin x\end{array}\right\} = \lim_{t\to\frac{1}{2}} \frac{2t^2 + t - 1}{2t^2 - 3t + 1} =$$

$2t^2 + t - 1 = 0$

$t_1 = -1;\ t_2 = \frac{1}{2}$

$2t^2 - 3t + 1 = 0$

$t_1 = 1;\ t_2 = \frac{1}{2}$

$$= \lim_{t\to\frac{1}{2}} \frac{2(t+1)(t-\frac{1}{2})}{2(t-1)(t-\frac{1}{2})} = \frac{1/2 + 1}{1/2 - 1} =$$

$$= \boxed{-3}$$

**2.163**

$$\lim_{x\to 0} \frac{\sin(x\sin 2x)}{x^2} = \lim_{x\to 0} \frac{x\cdot\sin 2x}{x^2} =$$

← $\delta_0$ $\sin x \sim x,\ x\to 0$

$$= \lim_{x\to 0} 2\cdot\frac{\sin 2x}{2x} = \boxed{2}$$

2. 153

$$\lim_{x\to 0}\frac{1-\cos x\cdot\cos 2x\cdot\cos 3x}{1-\cos x}=\left\{\frac{0}{0}\right\}=$$

$$=\lim_{x\to 0}\frac{(1-\cos x)+(\cos x-\cos x\cdot\cos 2x\cdot\cos 3x)}{1-\cos x}=$$

$$=\lim_{x\to 0}\left(1+\frac{\cos x\,(1-\cos 2x\cdot\cos 3x)}{1-\cos x}\right)=$$

$$=\lim_{x\to 0}\left(1+\frac{1\cdot\left((1-\cos 2x)+(\cos 2x-\cos 2x\cdot\cos 3x)\right)}{\frac{x^2}{2}}\right)=$$

$$=\lim_{x\to 0}\left(1+\frac{(2x)^2/2+\cos 2x\cdot(3x)^2/2}{x^2/2}\right)=1+\frac{4/2+9/2}{1/2}=$$

$$=\boxed{14}$$

# Домашнє завдання:

з 2.101 по 2.167 зробити будь-які 10 завдань з тих, що не робили